ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2007, Vol. 46, No. 1, pp. 78–82. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © M.Kh. Prilutskii, 2007, published in Izvestiya Akademii Nauk. Teoriya i Sistemy Upravleniya, 2007, No. 1, pp. 83–87.

Multicriterial Multi-Index Resource Scheduling Problems

M. Kh. Prilutskii
Nizhni Novgorod State University, pr. Gagarina 23, Nizhni Novgorod, 630600 Russia
Received July 4, 2006

Abstract—Resource scheduling problems are considered as multicriterial multi-index transport-type problems. Computational procedures for solving stated problems by using the direct search of an optimal node of a multidimensional multivalued cube are proposed.

DOI: 10.1134/S1064230707010091

INTRODUCTION

The wide class of decision making problems is related to the problem of the efficient planning and management of production systems. A typical representative of these problems is resource scheduling for units of an enterprise with a single-good or small-lot production. We consider a problem of this type not in the general form with all dependences and relations (as in [1, 2]), but with some degree of idealization. Instead of technological requirements, which are usually assigned by network structures, we consider stages of production and, instead of particular works with their duration, we take volume indices related to totalities of works involved at corresponding stages of production. Substantively, the resource scheduling problem is formulated as follows. It is necessary to distribute the general plan of an enterprise in volume characteristics (norm-hours, rubles, nominal tones, etc.) according to different indices: groups of equipment, periods of planning, stages of production, required resources, and types of production. Indices of the sought plan are divided into rigid ones, which require the obligatory fulfillment, and desirable ones, which are desired to be achieved. The rigid indices are formalized in the form of constraints, while the desirable ones are formalized in the form of optimality criteria. Then, the resource scheduling problem is stated as a multicriterial task (taking into account the desirable indices) with constrains (taking into account the rigid indices), which are linear in the considered idealization.

Assume that $i = \overline{1, m}$ are the numbers of the units of an enterprise, $j = \overline{1, n}$ are the numbers of orders, $k = \overline{1, s}$ are the numbers of products, and $t = \overline{1, q}$ are the numbers of the periods of planning. Denote by b_{ijkt} the volume of work that is outstanding in the ith unit for the jth order and the kth product at the period of planning t; by d_{it} the volume of work that can be done in the ith unit at the period of planning t; by r_{jk} the volume of work that must be done for the jth order and kth product; by v_{jkt} the obligatory volume of work at the period t for the j th order and kth product; and by w_j the necessary volume of work for the jth order, $i = \overline{1, m}$, $j = \overline{1, n}$, $k = \overline{1, s}$, $t = \overline{1, q}$.

Suppose that d_{it}, r_{jk}, and v_{jkt} are rigid and w_j are desirable indices of the sought plan. Then, formally, the resource scheduling problem consists in finding values of x_{ijkt} (the volume of work that will be done in the ith unit for the jth order and the kth product at the period of planning t), $i = \overline{1, m}$, $j = \overline{1, n}$, $k = \overline{1, s}$, $t = \overline{1, q}$) for which the following constraints hold:

$$\sum_{j=1}^{n}\sum_{k=1}^{s} x_{ijkt} \le d_{it}, \quad i = \overline{1, m}, \quad t = \overline{1, q}$$

(the volume of work that is implemented for all orders in the ith unit at the period of planning t must not exceed the power of this unit at the considered period of planning),

$$\sum_{i=1}^{m}\sum_{t=1}^{q} x_{ijkt} \ge r_{jk}, \quad j = \overline{1, n}, \quad k = \overline{1, s}$$

(the planned volume of work for the kth product of the j th order must be done in the units of the enterprise during the whole period of planning),

$$\sum_{i=1}^{m} x_{ijkt} \ge v_{jkt}, \quad j = \overline{1, n}, \quad k = \overline{1, s}, \quad t = \overline{1, q}$$

(the planned volume of work for the kth product of the j th order at the t th period of planning must be done in the units of the enterprise),

$$0 \le x_{ijkt} \le b_{ijkt},$$
$$i = \overline{1, m}, \quad j = \overline{1, n}, \quad k = \overline{1, s}, \quad t = \overline{1, q}$$

(natural constraints on the variables that formalize the requirements of the exclusion of unplanned production), taking into account the minimized criteria

$$f_j\left(\sum_{i=1}^{m}\sum_{k=1}^{s}\sum_{t=1}^{q}x_{ijkt}, w_j\right), \quad j = \overline{1, n},$$

given by functions that are nonnegative on each side of zero and take zero values at zero. These functions define the estimates of deviations of the sought indices from the desirable indices of the plan. For these functions, piecewise linear functions can be taken, while the convolution can be performed as a linear combination of these functions. Here, it is necessary for the formal statement of the problem that a user specifies the angles of inclination of the linear parts of the functions and convolution coefficients. However, as the experience of introducing has shown ([3, 4]), a user can only specify the boundaries of deviations, in which these deviations are "excellent", "very good", "good", "satisfactory", etc. Then, the functions of deviations $f_j(x)$ ($j = \overline{1, n}$) must be piecewise constant dividing the set of deviations for each criterion into domains of the "quality" of deviations. Such properties are possessed by functions with the range of values given by the set of positive nonnegative numbers from 0 to $p-1$ (0 means excellent, 1 means very good, etc.). Under various schemes of choosing rigid and desirable indices of the sought plan, resource scheduling problems are stated in different ways.

The problems in question possess the following specific features:

parameters of the mathematical model are multi-index and the number of indices can vary according to the considered problem;

the constraints of the mathematical model represent a system of linear algebraic transport-type inequalities, each obtained by summation over certain indices;

criteria of optimization problems are given in the form of step-functions, whose arguments are the sums of values of varied parameters over certain indices.

1. GENERAL MATHEMATICAL MODEL

In the most general case, the resource scheduling problem can be stated as follows. Boolean matrixes A and B of the dimensions $m \times k$ and $n \times k$, respectively, a real nonnegative vector $\vec{c}$ of the dimension m, and a vector function $\vec{F}(\vec{y})$ defined on the set of n-dimensional vectors from R^n with values from $\{0, 1, \dots, p-1\}$ are given. The introduced function $\vec{F}(\vec{y})$ maps the space R^n onto the set of nodes of an n-dimensional p-ary cube. It is required to find a vector $\vec{x}$ that satisfies the constraints $A\vec{x} \le \vec{c}$ taking into account the minimized criteria $\vec{F}(B\vec{x})$. The obtained problem is an n-criterion problem with linear constraints and partial optimality criteria whose structure is independent of the type of the function $\vec{F}(\vec{y})$. The system of the constraints $A\vec{x} \le \vec{c}$ is equivalent to the totality of linear algebraic transport-type equalities

$$\sum_{j \in A(i)} x_j \le c_i, \quad i = \overline{1, m},$$

here, $A(i)$ is the set of indices corresponding to the ith row of the Boolean matrix A ($i = \overline{1, m}$) and the vector $B\vec{x}$ has the coordinates

$$\sum_{j \in B(i)} x_j, \quad i = \overline{1, n},$$

where $B(i)$ is the set of indices corresponding to the ith row of the Boolean matrix B ($i = \overline{1, n}$). Relative to the vector function $\vec{F}(\vec{y})$, the following assumptions can be made.

For each node of the n-dimensional p-ary cube $\vec{z}$, the set of vectors $\vec{y}$ that satisfy the inequality $\vec{F}(\vec{y}) \le \vec{z}$ is a parallelepiped in the space R^k whose edges are parallel to the base vectors R^k.

If $\vec{z} = (p-1, p-1, \dots, p-1)$, then the set of vectors that are defined by the constraints $\vec{F}(\vec{y}) \le \vec{z}$ coincides with R^k.

We express the function $\vec{F}(\vec{y})$ as follows. For each component i, we consider the totality of the segments $S_i^{t_i}$, $S_i^{t_i} \subseteq S_i^{t_i+1}$, $t_i = \overline{0, p-2}$, $p \ge 1$, $i = \overline{1, n}$ embedded into one another. Then, $F_i\left(\sum_{j \in B(i)} x_j\right) = t_i$, if $\sum_{j \in B(i)} x_j \in S_i^{t_i}$, but $\sum_{j \in B(i)} x_j \notin S_i^{t_i-1}$. After the transformations performed, the resource scheduling problem is formulated as follows: it is required to find a vector $\vec{x}$ that satisfies the constraints $\sum_{j \in A(i)} x_j \le c_i$, $i = \overline{1, m}$ with account of the minimized criteria $F_i\left(\sum_{j \in B(i)} x_j\right) = t_i$, $t_i \in \{0, 1, \dots, p-1\}$, $i = \overline{1, n}$.

2. ALGORITHM OF THE SOLUTION

Reduce the stated multicriterial problem to the one-criterion problem by assigning onto the set of nodes of an n-dimensional p-ary cube of the linear order π, for which the following condition must hold: if for the nodes of the cube $\vec{\mu}$ and $\vec{\nu}$ defined by vectors of the space R^n the conditions $\vec{\mu} \ge \vec{\nu}$ are true (component by component), then $\vec{\mu}\,\pi\,\vec{\nu}$. This can be done, since the set of values of criteria (values of the vector function $\vec{F}(\vec{y})$) is a finite set. Then, the resource scheduling problem consists in finding a k-dimensional vector $\vec{x}^0$ involving the constraint $\sum_{j \in A(i)} x_j^0 \le c_i, i = \overline{1, m}$, for which at any $\vec{x}$ with the components $\sum_{j \in A(i)} x_j \le c_i, i = \overline{1, m}$ the relation $\vec{F}(B\vec{x}^0)\,\pi\,\vec{F}(B\vec{x})$ holds.

We assign to each node of the cube $\vec{z} = (z_1, z_2, \ldots, z_n)$ the system $S(\vec{z})$ of linear algebraic inequalities, which always includes the totality of the constraints $\sum_{j \in A(i)} x_j \le c_i, i = \overline{1, m}$ and, depending on the node of the cube, the set of two-sided constraints $\sum_{j \in B(i)} x_j \in S_i^{z_i}$, $i = \overline{1, n}$. On the set of the nodes of the cube, we assign the two-valued function $f(\vec{z})$, which takes the value 1 if the system $S(\vec{z})$ is compatible and 0, otherwise. The function $f(\vec{z})$ possesses the following property. If for the nodes of the cube $\vec{\mu}$ and $\vec{\nu}$, the inequality $\vec{\mu} \ge \vec{\nu}$ holds (componentwise), then $f(\vec{\mu}) \ge f(\vec{\nu})$, i.e., the function $f(\vec{z})$ is monotone.

Hence, to solve the resource scheduling problem, it is necessary to solve two problems: (1) finding an optimal node of a multidimensional multivalued cube, and (2) testing the consistency of systems of two-sided linear algebraic transport-type inequalities.

3. PROBLEM OF FINDING AN OPTIMAL NODE OF MULTIDIMENSIONAL MULTIVALUED CUBE (PROBLEM 1)

To solve the problem, the following computational procedure can be proposed. At the first step, the node $\vec{z}^* = (p-1, p-1, \ldots, p-1)$ of the cube and $f(\vec{z}^*)$ is computed. This is equivalent to testing the consistency the system $S(\vec{z}^*)$: $\sum_{j \in A(i)} x_j \le c_i, i = \overline{1, m}$. If the system is inconsistent, then the initial problem has no solution. Otherwise, a node $\vec{z}^1$ of the cube is chosen and $f(\vec{z}^1)$ is computed. This is equivalent to estimating the consistency of the system $S(\vec{z}^1)$. Depending on the value of $f(\vec{z}^1)$, the next node is taken, etc. The computational process must be finite. Among all nodes of the cube, an optimal one $\vec{z}^0$, for which $f(\vec{z}^0) = 1$, and $\vec{z}^0\,\pi\,\vec{z}$ at all $\vec{z}$, for which $f(\vec{z}) = 1$, is chosen. The solution of the initial resource scheduling problem is any admissible schedule of the consistent system $S(\vec{z}^0)$. As the order π defined on the set of nodes of the n-dimensional p-ary cube, we take the lexicographic relation; i.e., $\vec{z}^1\,\pi\,\vec{z}^2$ if and only if there exists i $(1 \le i \le n)$ such that, for the coordinates of the vectors $\vec{z}^1$ and $\vec{z}^2$, we have $z_k^1 = z_k^2$, $k = 1, 2, \ldots, i-1$ and $z_i^1 < z_i^2$. The algorithm for finding an optimal node of the cube in the lexicographic order for a monotone function consists of n steps. At the first step, among the nodes of the type $(z_1, p-1, p-1, \ldots, p-1)$, we find a value z_1^0, $(z_1^0 \in \{0, 1, \ldots, p-1\})$ that provides $f(z_1^0, p-1, \ldots, p-1) = 1$ and $z_1^0 \le z_1$ for all z_1, for which $f(z_1, p-1, \ldots, p-1) = 1$. At the second step, among the nodes of the type $(z_1^0, z_2, p-1, p-1, \ldots, p-1)$, we find analogously the second coordinate of the optimal node. At the nth step, we obtain the sought optimal node of the cube. The total number of computations of the function $f(\vec{z})$ has the order $n\log_2 p$.

Remark 1. In the case where the dimension of the cube is larger than the number of embedded segments, the described algorithm can be modified using binary search over the coordinates of the cube, rather than over values of coordinates. Then, an estimate for the number of computations of the function $f(\vec{z})$ has the order $p\log_2 n$.

4. TESTING THE CONSISTENCY OF TWO-SIDED LINEAR ALGEBRAIC SYSTEMS (PROBLEM 2)

Generally, to test the consistency of systems of linear algebraic inequalities $S(\vec{z})$, we can apply classical computational methods of linear algebra [5]. However, since the introduced constraints are of the transport type, it is possible to use special algorithms.

Generally, $S(\vec{z})$ can be represented as a system of linear differential transport-type inequalities with two-sided constraints

$$a_i \le \sum_{j \in Q(i)} x_j \le b_i, \quad i = \overline{1, q},$$

over the k-dimensional Euclidean space R^k. Here, $q = m + n$; $a_i = 0$ and $b_i = c_i$ if $i = \overline{1, m}$; a_i and b_i are the ends of the ith segment when $i = \overline{n+1, q}$; and $Q(i)$ is a set of indices for which the summation of the ith constraint is carried out $(i = \overline{1, q})$. If $\vec{x}^0$ is an arbitrary k-dimensional vector that satisfies all q constraints of the system, then the problem is solved. Assume that s is the first constraint in order whose conditions are violated.

Let us construct the vector $\vec{x}^1$ in accordance with the rule

$$x_j^1 = \begin{cases} x_j^0 + \left(a_s - \sum_{j \in Q(s)} x_j^0\right) \times |Q(s)|^{-1}, \text{ if } a_s > \sum_{j \in Q(s)} x_j^0, \\ x_j^0 - \left(\sum_{j \in Q(s)} x_j^0 - b_s\right) \times |Q(s)|^{-1}, \text{ if } \sum_{j \in Q(s)} x_j^0 > b_s, \end{cases} ,(j = \overline{1, k}),$$

and go to the $(s + 1)$th constraint. This is true for all constraints. If the system is consistent, then the consequence of the vectors $\vec{x}^\nu$ found by the described procedure, when $\nu \longrightarrow \infty$ converges to an admissible solution of the system. The proposed algorithm is the generalization of the Agmon–Motzkin relaxation method of orthogonal projections to the case of two-sided systems of algebraic inequalities of the transport type [6, 7].

Remark 2. As any iteration procedure, the proposed algorithm depends on two iteration parameters, namely, the accuracy ε (admissible violation of the inequalities of the system of constraints of problem 2 and the number of the steps of the algorithm h. We implemented the following scheme that makes it possible to solve actual problems in practice. In accordance with the computer performance, the value of h is chosen. Then, a knowingly large ε_0 is chosen and the problem 2is solved. If the solution is not found for h steps, then we make a hypothetical conclusion that the system of constraints in the problem is inconsistent. If the solution is found, then a new ε (e.g. $\varepsilon_0/2$) is taken. In such a way, using binary search, we compute a minimum value of ε, $\varepsilon \in [0, \varepsilon_0]$ for which the problem can be solved.

5. REDUCING PROBLEM 2 TO THE PROBLEM OF FINDING ADMISSIBLE CIRCULATION IN A TRANSPORT NETWORK

In the considered problems of resource scheduling, the varied parameters are multi-index and the constraints are represented by a system of linear algebraic transport-type inequalities, each of which is obtained by the summation over certain indices. Therefore, for convenience of notation, we can use the formalization proposed in [8] in the statement of multi-index transport linear programming problems. Assume that $N(s) = \{1, 2, \ldots, s\}$ and j_l is a certain parameter (index), $j_l \in J_l = \{1, 2, \ldots, n_l\}$, $l \in N(s)$; $f = \{k_1, k_2, \ldots, k_t\}$, $f \subseteq N(s)$. Denote by $F_f = (j_{k_1}, j_{k_2}, \ldots, j_{k_t})$ the set of values of indices, $F_f \in E_f$, $E_f = J_{k_1} \times J_{k_2} \times J_{k_t}$. We assign to each F_f a real number z_{F_f}, $F_f \in E_f$. The totality of these numbers for all possible values of indices $j_{k_1}, j_{k_2}, \ldots, j_{k_t}$ is represented as $\{z_{j_{k_1}, j_{k_2}, \ldots, j_{k_t}}\} = = \{z_{F_f}\}$. Let us introduce $\bar{f} = N(s)\backslash f$. Then, $F_{N(s)} = F_f F_{\bar{f}}$ designates the set $(k_1, k_2, \ldots, k_t, k_{t+1}, \ldots, k_s)$. Assume that

$$\sum_{F_f \in E_f} z_{F_f F_{\bar{f}}} = \sum_{j_{k_1} \in J_{k_1}} \sum_{j_{k_2} \in J_{k_2}} \cdots \sum_{j_{k_t} \in J_{k_t}} z_{F_f F_{\bar{f}}}, \quad F_{\bar{f}} \in E_{\bar{f}}.$$

As a result, the set of admissible solutions of the resource scheduling problem can be described as follows:

$$D(M) = \left\{ \{x_F\} | a_{F_{\bar{f}}} \le \sum_{F_f \in E_f} x_{F_f F_{\bar{f}}} \le b_{F_{\bar{f}}}, \quad F_{\bar{f}} \in E_{\bar{f}}, f \in M \right\},$$

here, M is the given set ($M \subseteq 2^{N(s)}$ and problem 2 consists in testing whether the set $D(M)$, $M \subseteq 2^{N(s)}$ is empty. Assume that $a_{F_{\bar{f}}}$ and $b_{F_{\bar{f}}}$ are integer numbers, $F_{\bar{f}} \in E_{\bar{f}}$, $f \in M$. Then, we can state, that the system of constraints that determines the set $D(M)$ is reduced to the problem of L-search for an admissible circulation in a transport network [9], if some subset of components of the solution of the problem L satisfies this system of constraints and, conversely, the system of constraints is inconsistent if the problem L does not have an admissible solution. Since the system of constraints that determined $D(M)$ depends on the set M ($M \subseteq 2^{N(s)}$, we find sets M such that the system of constraints is reduced to the problem L. It turns out [10] that, for this reduction, it is sufficient that a the partition $M_1 = \{f_1^{(1)}, f_2^{(1)}, \ldots, f_{m_1}^{(1)}\}$ and $M_2 = \{f_1^{(2)}, f_2^{(2)}, \ldots, f_{m_2}^{(1)}\}$ of the set M exists such that $f_i^{(1)} \subseteq f_{i+1}^{(1)}$, $i = \overline{1, m_1 - 1}$ and $f_i^{(2)} \subseteq f_{i+1}^{(2)}$, $i = \overline{1, m_2 - 1}$.

Hence, if for the set M this partition exists, then we can construct a transport network (with two-sided capacities of edges), in which an admissible circulation can be found by constructing the transport network corresponding to it (with one-sided capacitiesof edges) and solving for it the problem of finding a maximum flow, e.g., using the modified labeling algorithm [11]. If an admissible flow in a network with two-sided capacitiesof edges exists, then the corresponding system of constraints $D(M)$ is consistent. Since the computational complexity of the modified labeling algorithm has the order $O(|V|^3)$, where V is the set of nodes of the transport network, then the problem of consistency of systems of constraints of the type $D(M)$ is solved for $O(|V|^3)$ arithmetic operations.

Remark 3. For the resource scheduling problem presented in the Introduction, with account of the taken optimality criteria, $M = \{\{j, k\}, \{i, t\}, \{i\}, \{i, k, t\}, \varnothing\}$ there exists the partition $M_1 = \{\{i, k, t\}, \{i, t\}, \{i\}, \varnothing\}$, $M_2 = \{\{j, k\}\}$, for which the reducibility conditions hold.

Hence, to test the consistency of systems of the type $S(\overset{\geq}{z})$ corresponding to the considered problem, transport networks with two-sided capacitiesof edges can be constructed and problems for finding admissible flows in the constructed networks can be solved. At the last step of the algorithm, it is required to obtain a maximum flow, which describes an optimal solution of the initial problem.

REFERENCES

1. D. I. Batishchev, E. D. Gudman, I. P. Norenkov, et al., "A Decomposition Method for Solving Combinatorial Problems of Regulation and Resource Distribution," Informatsionnye Tekhnologii, No. 1 (1997).
2. D. I. Batishchev, E. D. Gudman, I. P. Norenkov, et al., "A Method of Combinatorial Heuristics for Solving Combinatorial Problems of Regulation and Resource Distribution," Informatsionnye Tekhnologii, No. 2 (1997).
3. M. Kh. Prilutskii and E. A. Kumagina, "The Problem of Regulation of Jobs as the Assignment Problem," Vestn. Nizhegor. Gos. Univ., Mat. Model. Opt. Upr., No. 21 (1999).
4. M. Kh. Prilutskii and S. E. Vlasov, "Optimal Resource Distribution in Problems of Scheduling and Structural Scheduling," Proceedings of Nizhegorodskii Gosudarstvennyi Tekhnicheskii Universitet: Data Processing and Control Systems, No. 11 (2004).
5. S. N. Chernikov, *Linear Inequalities* (Nauka, Moscow, 1968) [in Russian].
6. T. S. Motzkin and I. J. Schoenberg, "The Relaxation Method for Linear Inequalities," Caned. J. Moth **6** (3), 393–404 (1954).
7. M. Kh. Prilutskii, "Multicriterial Distribution of a Homogeneous Resource in Hierarchical Systems," Avtom. Telemekh. No. 2 (1996).
8. L. G. Raskin and I. O. Kirichenko, *Many-Index Linear Programming Problems* (Radio i Svyaz', Moscow, 1982) [in Russian].
9. L. R. Ford and D. R. Fulkerson, *Flows in Networks* (Princeton Univ. Press, Princeton 1962; Mir, Moscow, 1966).
10. M. Kh. Prilutskii and L. G. Afraimovich, "Consistency Conditions for Many-Index Transport Systems," Electronic Journal "Issledovano v Rossii", No. 70, 2005, http/www.Zhurnal.Are.Relarn.Ru/Artisles/25,7.
11. C. H. Papadimitriou, and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity* (Prentice-Hall, Englewood Cliffs, 1982; Mir, Moscow, 198).