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SYSTEM ANALYSIS AND OPERATIONS RESEARCH

Multi-index Transport Problems with Decomposition Structure

L. G. Afraimovich

Lobachevskii State University, Nizhni Novgorod, Russia

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Abstract—Consideration was given to the multi-index problems of linear and integer linear programming of the transport type. An approach based on the study of reducibility of the multi-index transport problems to that of seeking a flow on the network was proposed. For the multi-index problems with decomposition structure, a reduction scheme enabling one to solve the original multi-index problem using the cyclic decomposition of the minimum-cost flow of the auxiliary flow problem was constructed. The developed method underlies the heuristic algorithm to solve the NP-hard integer multi-index problem with a system of constraints featuring decompositional properties and general cost matrix.

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1. INTRODUCTION

There exists a wide class of applied problems of resource assignment that are formalizable as multi-index problems of (integer) linear programming of the transport type. They are represented (see [1–5]) by the problems of volume-calendar scheduling, assignment of the powers of data transmission channels, formation of the order portfolio, extraction and transportation of gas, processing of gas concentrate, and others. The multi-index assignment problems (subclass of the multi-index transport problems of integer linear programming) appear, for example, in the scheduling theory [6, 7] and computer vision [8, 9].

The general methods such as the simplex method or the Karmarkar algorithm [10, 11] can be used to solve the multi-index transport problems of linear programming. There exists a number of papers devoted directly to the methods of solving the multi-index problems of transport-type linear programming of which the best studied is the class of the two-index problems [12]. Special subclasses of the tri-index and four-index problems are considered, for example, in [13–15]. The general formulation of the class of multi-index problems is considered in [13]. The conditions under which one can reduce dimensionality and/or the number of indices of the multi-index transport problems were discussed in [16]. The geometrical properties of the set of permissible solutions of the multi-index transport systems of linear inequalities are discussed in [14, 17, 18].

Solution of the transport-type integer multi-index problems of linear programming is of special interest. The constraint matrix of the two-index transport problem is known to be absolutely unimodular, and, therefore, the class of two-index integer problems of linear programming is solvable in a polynomial time [12]. Yet the general formulation of the class of integer multi-index transport problems proves to be NP-hard even in the tri-index case [19]. Moreover, no polynomial ε-approximate algorithms exist for the problems of this class, otherwise, P = NP, this result being also applicable to the tri-index case [20]. In the absence of additional constraints on the parameters for solution of the multi-index integer problems, only the general methods of integer linear programming such as the branch-and-bound or Gomory methods [10, 21] which are exponential in terms of computational difficulty can be used to solve the multi-index integer problems. The class of multi-index assignment problems is best studied among the integer multi-index transport problems. An extensive review of the results obtained in the field of analysis of computational complexity and construction of the approximate algorithms to solve a special subclass of the multi-index assignment problem was given in [22]. Additionally, the papers [23–25] deserve mentioning.

Determination of the subclasses of problems solvable with the use of the flow methods represents one of the promising lines of development of efficient algorithms to study the multi-index linear programming problems. Active research in the area of network optimization [26] exerts great influence on its progress. For the case of reducibility of the linear programming problems to the flow ones, the existing efficient flow algorithms [27, 28] enable one to construct algorithms for their solution that have lower computational complexity as compared with the estimates established by the general methods for solution of the linear programming problems. In some cases, reduction to the flow problem also allows one to propose an algorithm guaranteeing determination of the integer solution of the original problem and thus to specify the polynomially solvable subclasses of problems among the problems of the integer linear programming. The possibility of reducing the linear programming problems to the flow problems was considered in [29–32] which from each other in the reducibility concepts used. The problem of reducibility of the multi-index transport problems of linear programming is less studied. The two-index problems are known to be reducible to the flow problems [12]. The question of reducibility of the multi-index problems with an arbitrary number of indices was considered in [1, 33, 34].

As was shown in [1, 33], the special conditions for two-embeddedness of the set of the subsets of indices over which summation is carried out in the problem constraint system are sufficient (necessary and sufficient in the case of tri-index problems; otherwise, P = NP) for reducibility the problem of minimal-cost flow. The reducibility concept used in [1, 33] is distinguished for the correspondence between the variables of the original problem and the arcs of the auxiliary network. The proposed scheme of reduction ensures that an arbitrary optimal flow of the auxiliary network defines an optimal solution of the original problem where the variables are assigned the values of flow along the corresponding arcs of the network. If the conditions for two-embeddedness are satisfied, then the solution of the transport-type multi-index linear programming problem comes to seeking a minimum-cost flow in the network with $O(m + n)$ vertices and $O(m + n)$ arcs, where n is the number of variables and m is the number of inequalities of the system of constraints of the original multi-index problem.

The concept of reduction of the system of linear inequalities to the problem of search of the permissible circulation was formulated in [34] at studying the multi-index systems of linear inequalities. Its important characteristic lies in the correspondence between the variables of the original system of inequalities and simple cycles of the auxiliary network. The proposed reduction scheme ensures that an arbitrary permissible circulation of the auxiliary network defines a permissible solution of the original inequality system where the values of flows along the corresponding simple cycles are assigned to the variables. The value of flow along the simple cycles is determined through the cyclic decomposition of the permissible circulation.

Various decomposition subclasses of the tri-index assignment problems were considered previously at studying the multi-index problems of integer linear programming. The criterion for the tri-index assignment problem is defined as $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}x_{ijk} \to \min$. The class of tri-index axial assignment problems with the cost matrix like $c_{ijk} = a_i b_j d_k,\ i, j, k = \overline{1, n}$, was studied in [35], and its NP-hardness was proved. Two classes of the tri-index axial assignment problems with the cost matrices like $c_{ijk} = a_{ij} + b_{jk} + d_{ki}$ and $c_{ijk} = \min(a_{ij} + b_{jk}, b_{jk} + d_{ki}, d_{ki} + a_{ij})$ were studied in [20], their NP-hardness was proved, and additionally they were proved to have no polynomial ε-approximate algorithms; otherwise, P = NP. Their NP-hardness was proved and the polynomial ε-approximate algorithms with $\varepsilon = 1/2$ and $\varepsilon = 1/3$, respectively, were constructed under the additional conditions of the triangle inequality for two decomposition classes considered in [20]. The multi-index transport problems with decomposition structure are less studied. The criterion in the multi-index transport problems is defined as $\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\dots\sum_{i_k=1}^{n} c_{i_1 i_2\dots i_k} x_{i_1 i_2\dots i_k} \to \min$. The multi-index axial assignment problems with the cost matrix having a decomposition structure like $c_{i_1 i_2\dots i_k} = f(d^{(1)}_{i_1 i_2}, d^{(2)}_{i_2 i_3}, \dots, d^{(k-1)}_{i_{k-1} i_k})$ were studied in [36], and the polynomial ε-approximate algorithms where ε is not a constant and depends on the parameters of the original problem were constructed. The estimates of ε are specified for certain classes of the function f. The results are summed up in [37] at studying the multi-index axial transport problems.

The present paper is a continuation of the studies of reducibility of the multi-index transport problem to the flow problems. In Section 2, the multi-index transport-type problems of linear programming are formalized, and the necessary notation is introduced. An auxiliary notion of cyclic decomposition of circulation that is used at describing the concept of reducibility is formulated in Section 3. Consideration is given to the algorithm to construct the cyclic decomposition. Section 4 is devoted to formalization of the reducibility concept which generalizes the scheme of reduction used in [34]. This scheme is grounded on the correspondence between the variables of the original problem and the simple cycles of the auxiliary network. Next, Section 5 shows that the special decomposition conditions for the set of the index subsets (defining the problem constraints) and the decomposition conditions of the multi-index cost matrix (defining the coefficient of the problem's objective function) suffice for reducibility to the problem of search of the minimum-cost flow. Under the established decomposition conditions, solution of the multi-index transport-type problem of linear programming comes to seeking a minimum-cost flow on a network with $O(n)$ vertices and $O(n)$ arcs, where n is the number of variables of the original multi-index problem. This method of solution of the multi-index problems with the decomposition structure is applicable also to the integer multi-index problems. The multi-index axial assignment problem (or the multi-index axial transport problem) with the cost matrix like $c_{i_i i_2\dots i_k} = d^{(1)}_{i_i i_2} + d^{(2)}_{i_2 i_3} + \dots + d^{(k-1)}_{i_{k-1} i_k}$ exemplifies the problem featuring the considered properties of decomposition. On the basis of the constructed method, Section 6 proposes a heuristic algorithm to solve the NP-hard class of the multi-index transport problems with a constraint system featuring properties of decomposition and the cost matrix of general form.

2. MULTI-INDEX TRANSPORT PROBLEMS

To formulate the multi-index transport-type problems of linear programming we make use of the formalization proposed in [13]. Let $s \in N$ and $N(s) = \{1, 2, \dots, s\}$. We assign to each number l a parameter j_l which is called the index and assumes values from the set $J_l = \{1, 2, \dots, n_l\}$, where $n_l \geqslant 2$, $l \in N(s)$. Let $f = \{k_1, k_2, \dots, k_t\} \subseteq N(s)$. The set of values of indices $F_f = (j_{k_1}, j_{k_2}, \dots, j_{k_t})$ will be called the t-index, and the set of all t-indices is defined as the Cartesian product of the sets of permissible values of the corresponding indices denoted by $E_f = J_{k_1} \times J_{k_2} \times \dots \times J_{k_t}$. Let $f' \subseteq f'' \subseteq N(s)$, where $f' = \{k'_1, k'_2, \dots, k'_{t'}\}$, $f'' = \{k''_1, k''_2, \dots, k''_{t''}\}$. Then, we denote $F_{f'} = (F_{f''})_{f'}$ if $F_{f'} = (j'_{k'_1}, j'_{k'_2}, \dots, j'_{k'_{t'}})$, $F_{f''} = (j''_{k''_1}, j''_{k''_2}, \dots, j''_{k''_{t''}})$ and $j'_{k'_i} = j''_{k'_i}$, $i = \overline{1, t'}$. If $F_{f'} \in E_{f'}$, $F_{f''} \in E_{f''}$, where $f', f'' \subseteq N(s)$ and $f' \cap f'' = \varnothing$, then we denote by $F_{f'}F_{f''}$ a set such that $F_{f'}F_{f''} \in E_{f' \cup f''}$ and $(F_{f'}F_{f''})_{f'} = F_{f'}$, $(F_{f'}F_{f''})_{f''} = F_{f''}$. Now, we determine $\overline{f} = N(s) \backslash f$. Then, according to the introduced notation $F_{N(s)} = F_f F_{\overline{f}}$ if $F_f = (F_{N(s)})_f$ and $F_{\overline{f}} = (F_{N(s)})_{\overline{f}}$.

A real number z_{F_f}, $F_f \in E_f$, is assigned to each set F_f. As in [13], this map of the set E_f of t-indices onto the set of real numbers is called the t-index matrix denoted by $\{z_{F_f}\}$. Let us consider the s-index matrix $\{z_{N(s)}\}$ using the following notation:

$$\sum_{F_f \in E_f} z_{F_f F_{\overline{f}}} = \sum_{j_{k_1} \in J_{k_1}} \sum_{j_{k_2} \in J_{k_2}} \dots \sum_{j_{k_t} \in J_{k_t}} z_{F_f\, F_{\overline{f}}}, \quad F_{\overline{f}} \in E_{\overline{f}}.$$

The notation of the subsums of the s-index matrix will be used to formalize the multi-index transport problems.

Let M be the given set, $M \subseteq 2^{N(s)}$; $\{a_{F_{\overline{f}}}\}$, $\{b_{F_{\overline{f}}}\}$, the given $|\overline{f}|$-index matrices of free coefficients, $0 \leqslant a_{F_{\overline{f}}} \leqslant b_{F_{\overline{f}}}$, $F_{\overline{f}} \in E_{\overline{f}}$, $f \in M$; $\{c_{F_{N(s)}}\}$, the given s-index matrix of the coefficients of the objective function; $\{x_{F_{N(s)}}\}$, the s-index matrix of the unknowns. Then, the multi-index transport-type problem of linear programming is formalized as follows:

$$a_{F_{\overline{f}}} \leqslant \sum_{F_f \in E_f} x_{F_f F_{\overline{f}}} \leqslant b_{F_{\overline{f}}}, \quad F_{\overline{f}} \in E_{\overline{f}}, \quad f \in M; \tag{1}$$

$$x_{F_{N(s)}} \geqslant 0, \quad F_{N(s)} \in E_{N(s)}; \tag{2}$$

$$\sum_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x_{F_{N(s)}} \to \min. \tag{3}$$

Problem (1)–(3) is denoted by $w(s; M; n_1, n_2, \ldots, n_s; \{a_{F_{\overline{f}}}\}, \{b_{F_{\overline{f}}}\}, f \in M; \{c_{F_{N(s)}}\})$, and under fixed M the class of all multi-index problems like (1)–(3) is denoted by $W(M)$. If $w \in W(M)$, then a constraint like (1) of the problem w corresponding to the fixed set $f \in M$ and fixed set $F_{\overline{f}} \in E_{\overline{f}}$ is denoted by $d(w, f, F_{\overline{f}})$.

3. CYCLIC DECOMPOSITION OF FLOW

Search of the minimum-cost circulation on the network with bilateral throughputs is considered as the flow problem. It is common knowledge that the permissible circulation can be established using the algorithm for search of the maximal flow in the corresponding canonical network, and the minimum-cost circulation, using the algorithm to seek the minimum-cost flow of the predefined value [26]. Importantly, the constraint system matrix of the problem of seeking the circulation is an absolutely unimodular one [10]. Therefore, in the joint network with the integer throughputs there always exists an integer permissible circulation. It can be decomposed into flows along the simple cycles of the network. The integer circulation can be decomposed into integer flows along cycles. These results will be used in what follows to study reducibility of the multi-index transport problems to the flow problems.

Let us consider a formulation of the problem of seeking a minimum-cost flow (circulation) [26]. Let $G = (V_G, A_G)$ be an oriented loop-free graph. Here, V_G and A_G are, respectively, the sets of vertices and arcs of the graph G, $A_G \subseteq V_G^2$. The subscript G in the notation of the sets of vertices and arcs is omitted wherever this does not create ambiguity. We denote by l_{ij} and u_{ij}, respectively, the lower and upper throughputs of the arc (i, j), $0 \leqslant l_{ij} \leqslant u_{ij}$; e_{ij} is the cost of the arc (i, j); x_{ij} is the magnitude of the flow through the arc (i, j), $(i, j) \in A$. It is required to determine the unknown variables x_{ij}, $(i, j) \in A$, which are solutions of the following linear programming problem:

$$\sum_{(j,i) \in A} x_{ji} - \sum_{(i,j) \in A} x_{ij} = 0, \quad i \in V; \tag{4}$$

$$l_{ij} \leqslant x_{ij} \leqslant u_{ij}, \quad (i, j) \in A; \tag{5}$$

$$\sum_{(i,j) \in A} e_{ij} x_{ij} \to \min. \tag{6}$$

Problem (4)–(6) is denoted below by $v(G; l_{ij}, u_{ij}, e_{ij}, (i, j) \in A)$. By the circulation is meant a set of nonnegative values x_{ij}, $(i, j) \in A$, satisfying the constraint system (4). If additionally $x_{ij} \in Z$, $(i, j) \in A$, then by the circulation is meant an integer. By the permissible circulation is meant a set of values x_{ij}, $(i, j) \in A$, satisfying the constraint system (4), (5). By the minimum-cost circulation is meant the set of values of x_{ij}, $(i, j) \in A$, which is the solution of problem (4)–(6).

The simple cycle of the graph G is denoted below for convenience by $C = (i_1, i_2, \ldots, i_{k+1})$, where $(i_j, i_{j+1}) \in A$, $j = \overline{1,k}$; $i_1 = i_{k+1}$; $i_{j'} \neq i_{j''}$ for $j' \neq j''$, $j', j'' = \overline{1,k}$. We also assume for definiteness that $i_1 < i_j$, $j = \overline{2,k}$. Let $(u,v) \in A$, $C = (i_1, i_2, \ldots, i_{k+1})$. Then, the notation $(u,v) \in C$ means that there exists $j \in \{1, 2, \ldots, k\}$ such that $u = i_j$, $v = i_{j+1}$; $(u,v) \notin C$ denotes the inverse. One can readily see that the number of simple cycles in the graph is upper-bounded, for example, by $\sum\limits_{i=1}^{|V|} C_{|V|}^{i}(i-1)!$ and, therefore, the set of simple cycles in the graph is finite. The set of all simple problems of the graph G is denoted by $C(G)$.

Definition 1. Let x_{ij}, $(i,j) \in A$ be the circulation of the graph G. By the cyclic decomposition of circulation is meant a set of nonnegative values y_C, $C \in C(G)$, such that $\sum\limits_{C \in C(G)|(i,j) \in C} y_C = x_{ij}$, $(i,j) \in A$. If additionally $y_C \in Z$, $C \in C(G)$, then the cyclic decomposition is called the integer decomposition.

The following results are known.

Assertion 1 [34]. *For an arbitrary circulation, there exists its cyclic decomposition. For an arbitrary integer circulation, there exists its integer cyclic decomposition.*

Assertion 2 [34]. *There exists an algorithm to construct a cyclic decomposition from the given circulation which requires $O(|V||A|)$ computing operations.*

It deserves noting that the algorithm to construct the cyclic decomposition that was proposed in [34] guarantees construction of an integer cyclic decomposition from the given integer circulation.

4. CONCEPT OF REDUCIBILITY

We present a formalization of the reducibility concept which will be used in what follows to study the reducibility of the multi-index problems to the flow problems. Let $A \in R^{n \times m}$, $b, b^-, b^+ \in R^n$, $c \in R^m$ be the given parameters, and $x \in R^m$ be vector of unknown variables. We denote by $w(A, b, c)$ the linear programming problem $\min\{(c,x)|Ax \leqslant b,\ x \geqslant 0\}$, and by $w(A, b^-, b^+, c)$, the linear programming problem $\min\{(c,x)|b^- \leqslant Ax \leqslant b^+,\ x \geqslant 0\}$. The numbers of rows and columns of the matrix A are denoted for convenience by $row(A)$ and $col(A)$, respectively. We notice that the problem $w(A, b^-, b^+, c)$ can be described using notation like $w(A, b, c)$. Nevertheless, we use the notation $w(A, b^-, b^+, c)$ if we want to emphasize that the constraint system is a system of two-sided inequalities. We also consider the problems of integer linear programming. If $w = w(A, b, c)$ is a linear programming problem, then we denote by w_Z the problem of integer linear programming $w_Z = \min\{(c,x)|Ax \leqslant b, x \in Z_+^{col(A)}\}$. Let W be an arbitrary class of linear programming problems. The corresponding class of problems of integer linear programming is defined as $W_Z = \{w_Z | w \in W\}$.

We consider two classes W' and W'' of the linear programming problems. By the reducibility of the class W' to the class W'' is meant the possibility of constructing for the arbitrary problem $w' \in W'$ of the corresponding problem $w'' \in W''$ so that the solution of w'' defines solution of w'. At formalizing a particular reduction scheme, we determine the time costs and/or particular computing procedures related with

—construction of the constraint system matrix of problem w'' from the original parameters of problem w';

—construction of the free coefficients and the coefficients of the objective function of problem w'' from the original parameters of problem w';

—construction of the solution of problem w' from the solution of problem w''.

The notation of the reduction scheme as suggested below is introduced by analogy with the notation of R. Graham used to classify the problem of the scheduling theory [38].

Definition 2. The class W' will be said to be $t_1 - s_1|t_2 - s_2|t_3 - s_3$ reducible to the class W'' if for any problem $w' = w(A', b', c') \in W'$ it is possible to construct the matrix A'' in time $O(t_1)$ and in time $O(t_2)$ the vectors b'' and c'' such that $w'' = w(A'', b'', c'') \in W''$, and at that

—the problem w' is consistent (bounded) if and only if the problem w'' is consistent (bounded);

—if the optimal (permissible) solution x'' of problem w'' is known, then the optimal (permissible) solution x' of the problem w' can be constructed in time $O(t_3)$.

Here, $(-s_1)$, $(-s_2)$, $(-s_3)$ is the optional row notation of the computing procedures related with construction of the constraint system matrix, free coefficient, and coefficients of the objective function and with construction of the problem solution, correspondingly.

By the problem w'' (see Definition 2) is meant a problem corresponding to w'. Sometimes, the estimators of the computing complexity t_1, t_2, and t_3 are denoted for convenience by L or P by which are meant functions depending linearly or polynomially on the size of the particular problem w'.

The present paper considers the possibility of reducing the subclass of multi-index transport-type linear programming problems to the class of seeking the minimum-cost circulation. We consider at that the reduction schemes where solution of the original multi-index problem is determined through the cyclic decomposition minimum-cost flow of the corresponding flow problem. The class of flow problems is defined as follows. We denote by *Graph* the set of all oriented loop-free graphs. The class of search of the minimum-cost flow is defined as the $W_{Graph} = \{v(G, l_{ij}, u_{ij}, e_{ij}, (i,j) \in A_G)|l_{ij}, u_{ij} \in R_+, e_{ij} \in R, (i,j) \in A_G, G \in Graph\}$.

Definition 3. Let W be the class of linear programming problems with two-sided system of inequalities. The class W is said to be $t_1|t_2 - equal|t_3 - cycle$ reducible to the class W_{Graph} if the class W is $t_1|t_2|t_3$ reducible to the class W_{Graph}; and if $v = v(G; l_{ij}, u_{ij}, e_{ij}, (i,j) \in A_G) \in W_{Graph}$ is a problem corresponding to the problem $w = w(A, b^-, b^+, c) \in W$, then there exist injective functions $\alpha : \{1, 2, \ldots, row(A)\} \to A_G, \beta : \{1, 2, \ldots, col(A)\} \to C(G)$ such that the following conditions are satisfied:

—$l_{\alpha(i)} = b_i^-$, $u_{\alpha(i)} = b_i^+$, $i = \overline{1, row(A)}$; $l_{(u,v)} = 0, u_{uv} = b^*$, $(u,v) \in A_G \backslash \{\alpha(i)|i = \overline{1, row(A)}\}$, where b^* is some sufficiently great value,—for definiteness, $b^* = \sum\limits_{i=1}^{row(A)} b_i^+$;

—if the circulation z_{ij}, $(i,j) \in A_G$, is the optimal (permissible) solution of the problem v and y_C, $C \in C(G)$, is the cyclic decomposition of the given circulation, then $x = (y_{\beta(1)}, y_{\beta(2)}, \ldots, y_{\beta(col(A))})$ is the optimal (permissible) solution of the problem w.

Therefore, according to Definition 3, in the case of $t_1|t_2 - equal|t_3 - cycle$ reducibility of the class W to the class W_{Graph} it is ensured that if $w \in W$, $v = v(G; l_{ij}, u_{ij}, c_{ij}, (i,j) \in A_G) \in W_{Graph}$ and v is a problem corresponding to the problem w, then at constructing the problem of search of the minimum-cost flow v the throughputs and costs of arcs are determined in the problem in terms of the coefficients of the problem w, and the solution of the problem w is determined through the cyclic decomposition of the solution of the problem v. Then, an algorithm of the computing complexity $O(t_1 + t_2 + t_3 + \mu(|V_G|, |A_G|))$, where $\mu(n, m)$ is the computing complexity of the algorithm for solution of the problem of search of the minimum-cost flow in a network with n vertices and m arcs, can be suggested to solve the problem w on the basis of the solution of the corresponding problem v. A review of the estimates of the computing complexity for the existing flow algorithms can be found, for example, in [27, 28].

Theorem 1. *Let the class of problems W be $P|P - equal|P - cycle$ reducible to the class W_{Graph}. Then, the class of problems of integer linear programming W_Z is solvable in a polynomial time.*

The proofs of Theorem 1 and the following Lemmas 1 and 2 and Theorems 2 and 3 are given in the Appendix. According to Theorem 1, separation of the subclasses of the multi-index problems reducible to the flow problems also allows one to separate the polynomially solvable subclasses in the NP-hard class of the integer multi-index problems.

5. REDUCIBILITY CONDITIONS FOR THE MULTI-INDEX PROBLEMS

We consider now the questions of constructing the subclasses of the multi-index problems which can be reduced to the flow problems using the concept introduced in Definition 3. The class of multi-index transport problems with a special decomposition structure turns out to be one of such subclasses.

Definition 4. Let $M \subseteq 2^{N(s)}$ and $f_1, f_2, \ldots, f_k$ be the decomposition of the set $N(s)$. It will be said that M is an $f_1, f_2, \ldots, f_k$-decomposition if $M \subseteq \{\overline{f_i} | i = \overline{1, k}\} \cup \{\overline{f_i \cup f_{i+1}} | i = \overline{1, k-1}\}$.

Definition 5. Let $\{c_{F_{N(s)}}\}$ be the s-index matrix and $f_1, f_2, \ldots, f_k$, the decomposition of the set $N(s)$. The multi-index matrix $\{c_{F_{N(s)}}\}$ will be said to be the $f_1, f_2, \ldots, f_k$-decomposition if there are multi-index matrices $\{d_{F_{f_i}}\}$ $f \in B$ such that $c_{F_{N(s)}} = \sum\limits_{f \in B} d_{(F_{N(s)})_f}$, where $B = \{f_i | i = \overline{1,\, k}\} \cup \{f_i \cup f_{i+1} | i = \overline{1,\, k-1}\}$.

Definition 6. Let $f_1, f_2, \ldots, f_k$ be the decomposition of the set $N(s)$. Then, we denote by $W^D(f_1, f_2, \ldots, f_k)$ the following class of the multi-index transport problems with the decomposition structure:

$$W^D(f_1, f_2, \ldots, f_k) = \Big\{ w(s; M; n_1, n_2, \ldots, n_s; \{a_{F_{\overline{f}}}\}, \{b_{F_{\overline{f}}}\}, f \in M; \{c_{F_{N(s)}}\}) | \\ M \text{ and } \{c_{F_{N(s)}}\} \text{ are } f_1, f_2, \ldots, f_k\text{-decomposition} \Big\}.$$

Lemma 1. *Let $f_1, f_2, \ldots, f_k$ be the decomposition of the set $N(s)$. Then,*

$$\sum_{i=1}^{k} |E_{f_i}| \leqslant |E_{N(s)}|.$$

Lemma 2. *Let $f_1, f_2, \ldots, f_k$ be the decomposition of the set $N(s)$. Then,*

$$\sum_{i=1}^{k-1} |E_{f_i}||E_{f_{i+1}}| \leqslant |E_{N(s)}|.$$

Theorem 2. *Let $f_1, f_2, \ldots, f_k$ be decomposition of the set $N(s)$. Then, the class of problems $W^D(f_1, f_2, \ldots, f_k)$ is $L|L - equal||E_{N(s)}|^2 - cycle$ reducible to the class W_{Graph}.*

The constructive scheme of the proof of Theorem 2 suggests an algorithm to solve problems of the class $W^D(f_1, f_2, \ldots, f_k)$ by constructing the corresponding flow problem, seeking the minimum-cost flow, and solving the original multi-index problem by cyclic decomposition of the established flow. The well-known flow algorithms [27, 28] may be used to seek the minimum-cost flow. Whence the following corollary is obtained.

Corollary 1. *Let $f_1, f_2, \ldots, f_k$ be a decomposition of the set $N(s)$. Then, there exists an algorithm to solve problems of the class $W^D(f_1, f_2, \ldots, f_k)$ requiring $O(|E_{N(s)}|^3 \log^2 |E_{N(s)}|)$ computing operations.*

The following result is obtained directly from Theorems 1 and 2 and Corollary 1.

Corollary 2. *Let $f_1, f_2, \ldots, f_k$ be the decomposition of the set $N(s)$. Then, the class of problems of integer linear programming $W^D_Z(f_1, f_2, \ldots, f_k)$ is solvable in a polynomial time.*

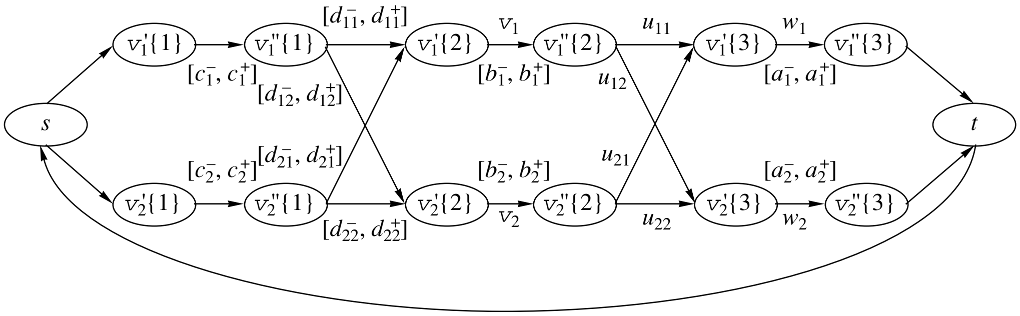

Figure.

We present an example of the multi-index transport problem with decomposition structure which is solved using the above approach. Let $s = 3$. We consider the multi-index transport-type problem of linear programming:

$$a_{j_3}^- \leqslant \sum_{j_1 \in J_1} \sum_{j_2 \in J_2} x_{j_1 j_2 j_3} \leqslant a_{j_3}^+, \quad j_3 \in J_3;$$

$$b_{j_2}^- \leqslant \sum_{j_1 \in J_1} \sum_{j_3 \in J_3} x_{j_1 j_2 j_3} \leqslant b_{j_2}^+, \quad j_2 \in J_2;$$

$$c_{j_1}^- \leqslant \sum_{j_2 \in J_2} \sum_{j_3 \in J_3} x_{j_1 j_2 j_3} \leqslant c_{j_1}^+, \quad j_1 \in J_1;$$

$$d_{j_1 j_2}^- \leqslant \sum_{j_3 \in J_3} x_{j_1 j_2 j_3} \leqslant d_{j_1 j_2}^+, \quad j_3 \in J_3;$$

$$x_{j_1 j_2 j_3} \geqslant 0, \quad j_1 \in J_1, \quad j_2 \in J_2, \quad j_3 \in J_3;$$

$$\sum_{j_1 \in J_1} \sum_{j_2 \in J_2} \sum_{j_3 \in J_3} (u_{j_2 j_3} + v_{j_2} + w_{j_3}) x_{j_1 j_2 j_3} \to \min .$$

This multi-index problem is belongs to the class $W(M)$ where $M = \{\{1,2\},\{1,3\},\{2,3\},\{3\}\}$. We take the decomposition $f_1 = \{1\},\ f_2 = \{2\},\ f_3 = \{3\}$ of the set $N(3)$. One can readily see that $M \subseteq \{f_i | i = \overline{1,3}\} \cup \{f_i \cup f_{i+1} | i = \overline{1,2}\} = \{\{1,2\},\{1,3\},\{2,3\},\{1\},\{3\}\}$. Therefore, according to Definition 4 M is an f_1, f_2, f_3-decomposition set. Now, we notice that $\{\{2,3\},\{2\},\{3\}\} \subseteq B = \{f_i | i = \overline{1,3}\} \cup \{f_i \cup f_{i+1} | i = \overline{1,2}\} = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$. Whence it follows that, according to Definition 5, the multi-index cost matrix of the problem (as defined by $u_{j_2 j_3} + v_{j_2} + w_{j_3}$) is the f_1, f_2, f_3-decomposition one. Consequently, according to Definition 6, the multi-index problem under consideration belongs to the class $W^D(f_1, f_2, f_3)$, $L|L - equal||E_{N(s)}|^2 - cycle$ reducible to the class W_{Graph}, where $|E_{N(s)}| = |J_1||J_2||J_3|$.

Now we consider an example of the reduction scheme used in the constructive proof of Theorem 2 (see the Appendix). Let $|J_1| = |J_2| = |J_3| = 2$. We present the transport network defining the problem of seeking the minimum-cost flow and corresponding to the original multi-index problem.

For some arcs the figure shows their throughputs and costs. The arcs without indication of the throughput segments have zero lower throughput and unlimited upper throughput. The arcs without indication of cost have zero cost. According to the proof of Theorem 2, the simple cycle $(s, v_{j_1}'\{1\}, v_{j_1}''\{1\}, v_{j_2}'\{2\}, v_{j_2}''\{2\}, v_{j_3}'\{3\}, v_{j_3}''\{3\}, t, s)$, $j_1 \in J_1$, $j_2 \in J_2$, $j_3 \in J_3$ is assigned to the variable $x_{i_1 i_2 i_3}$. In virtue of $L|L - equal||E_{N(s)}|^2 - cycle$ reducibility of the class $W^D(f_1, f_2, f_3)$ to the class W_{Graph}, solution of the original multi-index problem is defined as follows: each variable of the multi-index problem is assigned a value of the flow along the corresponding simple cycle which in turn is obtained as a result of cyclic decomposition of the minimum-cost flow of the given network.

6. APPROXIMATE SOLUTION OF THE INTEGER MULTI-INDEX PROBLEMS

The class of the multi-index problems of integer linear programming is NP-hard [19]. It is also known that in the general case there exist no approximate polynomial algorithms for the multi-index problems of integer linear programming; otherwise, P = NP [20]. Additionally, it may be shown that there are no efficient approximate algorithms even in the class of the integer multi-index problems with a system of constraints having the decomposition properties and the cost matrix of general form.

The ε-approximate algorithm of [19] is as follows. Let W be the class of minimization problems; H, the algorithm which returns the permissible solution $H(w)$ of the problem w for any problem $w \in W$; $c(H(w))$, the value of the criterion of the problem w on the permissible solution $H(w)$; and $OPT(w)$, optimal value of the criterion of w. Then, the algorithm H is called the ε-approximate algorithm for the problems of class W, where ε is a nonnegative constant if the condition $c(H(w)) \leqslant (1+\varepsilon)OPT(w)$, $w \in W$, is met.

Theorem 3. *Let $f_1, f_2, \ldots, f_k$ be the decomposition of the set $N(s)$ and $k \geqslant 3$. Then, there exists $f_1, f_2, \ldots, f_k$-decomposition set M such that for the problems of the class $W_Z(M)$ there exists no polynomial ε-approximate algorithm for any $\varepsilon \geqslant 0$; otherwise,* P = NP.

The following approach is proposed for seeking an approximate solution of the NP-hard problems with a constraint system featuring the decomposition properties and having the general cost matrix. We seek a multi-index cost matrix having the desired properties of decomposition and being the "closest" to the cost matrix of the original problem. Then we determine solution of the auxiliary problem with the original system of constraints and a new cost matrix. To seek solution of the auxiliary problem, one may use the polynomial algorithm for solution of the problems of the class $W_Z^D(f_1, f_2, \ldots, f_k)$ (see Corollaries 1 and 2). The established solution is a permissible solution of the original multi-index problem of integer linear programming. By substituting the determined solution of the auxiliary problem in the criterion of the original problem, we get a reachable estimate that can be used, for example, for solving the NP-hard multi-index problems by the branch-and-bound method [11].

Let us consider the proposed approach by way of example of the well-known NP-hard axial tri-index assignment problem [19] which is formulated as the following Boolean programming problem:

$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} x_{j_1j_2j_3} = 1, \quad j_3 = \overline{1,n}; \tag{7}$$

$$\sum_{j_1=1}^{n}\sum_{j_3=1}^{n} x_{j_1j_2j_3} = 1, \quad j_2 = \overline{1,n}; \tag{8}$$

$$\sum_{j_2=1}^{n}\sum_{j_3=1}^{n} x_{j_1j_2j_3} = 1, \quad j_1 = \overline{1,n}; \tag{9}$$

$$x_{j_1j_2j_3} \in \{0,\ 1\}, \quad j_1, j_2, j_3 = \overline{1,n}; \tag{10}$$

$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n} c_{j_1j_2j_3} x_{j_1j_2j_3} \to \min. \tag{11}$$

Condition (10) is replaced by

$$x_{j_1j_2j_3} \in Z_+, \quad j_1, j_2, j_3 = \overline{1,n}. \tag{12}$$

One can readily see that problems (7)–(11) and (7)–(9), (12), (11) are equivalent. Let now $s = 3$ and $M = \{\{1,\ 2\},\ \{2,\ 3\},\ \{1,\ 3\}\}$. Problem (7)–(9), (12), (11) belongs to the class $W_Z(M)$. We notice that $N(3) = \{1,\ 2,\ 3\}$ and $\{1\}, \{2\}, \{3\}$ is a decomposition of the set $N(3)$. According to Definition 4, M is the $\{1\}, \{2\}, \{3\}$-decomposition set. However, it follows from Theorem 1 and NP-hardness of the class $W_Z(M)$ that the class of problems $W(M)$ is not $P|P-equal|P-cycle$ reducible to the class W_{Graph}. Moreover, for the class $W_Z(M)$ there exists no polynomial ε-approximate algorithm for any $\varepsilon \geqslant 0$, otherwise P = NP [20]. On the other hand, there are among the problems of the class $W(M)$ those whose multi-index cost matrix of the objective function has decomposition properties. Let us consider the tri-index matrix $e_{j_1j_2j_3} = u_{j_1j_2} + v_{j_2j_3}$, $j_1, j_2, j_3 = \overline{1,n}$ where $\{u_{j_1j_2}\}$ and $\{v_{j_2j_3}\}$ are two-index matrices. According to Definition 5, the tri-index matrix $\{e_{j_1j_2j_3}\}$ is the $\{1\},\{2\},\{3\}$-decomposition one. Let us consider the criterion

$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n}(u_{j_1j_2}+v_{j_2j_3})x_{j_1j_2j_3} \to \min\,. \tag{13}$$

The subclass of problems $W(M)$ where the objective function is representable as (13) is included in the class $W^D(\{1\},\{2\},\{3\})$ and according to Theorem 2 is $L|L-equal||E_{N(s)}|^2-cycle$ reducible to the class W_{Graph}. Whence it follows that, according to Corollary 2, problem (7)–(9), (12), (13) is solvable in a polynomial time. We notice that here $|E_{N(s)}| = n^3$.

In the general case, at solving problems (7)–(9), (12), (11) we seek a $\{1\},\{2\},\{3\}$-decomposition matrix like $\{e_{j_1j_2j_3}\}$ which is most "close" to the original general-form matrix $\{c_{j_1j_2j_3}\}$. We consider the following problem of search of the nearest decomposition of the cost matrix:

$$e_{j_1j_2j_3} = u_{j_1j_2} + v_{j_2j_3}, \quad j_1, j_2, j_3 = \overline{1,n}; \tag{14}$$

$$\operatorname{dist}(\{c_{j_1j_2j_3}\},\{e_{j_1j_2j_3}\}) \to \min, \tag{15}$$

where $e_{j_1j_2j_3},\ u_{j_1j_2},\ v_{j_2j_3}, j_1, j_2, j_3 = \overline{1,\ n}$, are the real unknown variables, and dist is some measure of closeness of the multi-index matrices. Let us consider the function of the squared Euclidean distance as the dist function. Then, problem (14), (15) becomes that of the unconditional quadratic optimization:

$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n}(c_{j_1j_2j_3}-u_{j_1j_2}-v_{j_2j_3})^2 \to \min\,. \tag{16}$$

The methods of minimization of the sum of squared residues may be used for solution of problem (16) [39]. Let $u^*_{j_1j_2},\ v^*_{j_2j_3},\ j_1, j_2, j_3 = \overline{1,\ n}$, be solution of problem (16). Then, instead of the original problem (7)–(9), (12), (11), we solve the problem with the same system of constraints and with the criterion $\sum\limits_{j_1=1}^{n}\sum\limits_{j_2=1}^{n}\sum\limits_{j_3=1}^{n}(u^*_{j_1j_2}+v^*_{j_2j_3})x_{j_1j_2j_3} \to \min$. According to Corollary 2, solution of the constructed problem can be established in a polynomial time because it belongs to the class $W_Z^D(\{1\},\{2\},\{3\})$. As was already noted, the established solution is a permissible solution of the original NP-hard assignment problem which may be used, for example, to seek the reachable estimate at using the branch-and-bound method.

7. CONCLUSIONS

The present paper studies the $t_1|t_2-equal|t_3-cycle$ reducibility of the multi-index transport problems to the class of problems of seeking the minimum-cost flow. The considered concept of reduction enables one to establish correspondence between the variables of the original problem and simple cycles of the auxiliary network. At that, the minimum-cost flow of the auxiliary network defines an optimal solution of the original problem such where the variables are assigned the values of flow along the corresponding simple cycles. The value of flow along simple cycles is determined using the cyclic decomposition of the flow.

Specified was the class of multi-index transport problems of the decomposition structure for which the $L|L-equal||E_{N(s)}|^2-cycle$ reducibility to the class of problems of seeking the minimum-cost flow was proved. Relying on reducibility, constructed was a polynomial algorithm to solve the multi-index transport problems with decomposition structure. The algorithm requires

$O(|E_{N(s)}|^3 \log^2 |E_{N(s)}|)$ computing operations, where $|E_{N(s)}|$ is equal to the number of variables of the multi-index problem. The constructed algorithm is also applicable to seeking an integer solution.

The polynomial solvability of the integer multi-index transport problems with the decomposition structure is of special interest for the following reasons. Formalization of the established class of problems with the decomposition structure is related to some decomposition characteristics of the constraint system of the problem and its multi-index matrix of criterion costs. As it happens, for the class of the integer multi-index transport problems with a constraint system featuring the decomposition characteristics and general cost matrix there exist no polynomial ε-approximate algorithms for any $\varepsilon \geqslant 0$; otherwise, P = NP. Whence it follows that for the constraint systems of the considered NP-hard problems of integer linear programming the specified class of the multi-index cost matrices having the decomposition properties enables one to determine the class of criteria for which the problem becomes polynomially solvable. On the basis of the constructed polynomial algorithm for solution of the multi-index transport problems with the decomposition structure, a heuristic algorithm to solve the NP-hard difficult-to-approximate integer multi-index problems with a constraint system featuring the decomposition characteristics was proposed. It seeks a matrix with the decomposition characteristics which is "closest" to the original cost matrix.

Further line of research is oriented to determining the necessary and sufficient conditions for the $t_1|t_2 - equal|t_3 - cycle$ reducibility for the multi-index transport problems and developing new concepts of reducibility enabling extension of the application domain of the flow algorithms for solution of the multi-index problems. Estimation of deviations of the proposed heuristic algorithm from the optimum is of special interest.

APPENDIX

Proof of Theorem 1. Let the class W be $P|P - equal|P - cycle$ reducible to the class W_{Graph}. Then, according to Definition 3, for an arbitrary problem $w \in W$ there exists its corresponding problem $v \in W_{Graph}$. At that, the corresponding problem v can be constructed in a polynomial time dependent on the dimension of the problem w. Consequently, the dimension of the problem v also depends polynomially on the dimension of the original problem w. The integer solution of the flow problem v is known to be determinable by using, for example, the polynomial methods of solution of the flow problems [26]. According to Assertions 1 and 2, the integer cyclic decomposition of the integer solution of the problem v can be constructed in a polynomial time. According to Definition 3, the integer solution of the problem w can be determined in a polynomial time in terms of the constructed integer cyclic decomposition. Then, the integer solution of the problem w determined in a polynomial time is also a solution of the problem w_Z. Whence it follows that the class of problems W_Z is solvable in a polynomial time, which proves Theorem 1.

Proof of Lemma 1. Let $f_1, f_2, \ldots, f_k$ be a decomposition of the set $N(s)$. Then, $|E_{f_i}| = \prod\limits_{j \in f_i} n_j$, $i = \overline{1,k}$, and $|E_{N(s)}| = \prod\limits_{l=1}^{s} n_l = \prod\limits_{i=1}^{k} \prod\limits_{j \in f_i} n_j = \prod\limits_{i=1}^{k} |E_{f_i}|$. We denote for convenience $|E_{f_i}| = m_i$, $i = \overline{1,k}$. Then, $|E_{N(s)}| = \prod\limits_{i=1}^{k} m_i$. Since $n_l \geqslant 2$, $l = \overline{1,s}$, we get $m_i \geqslant 2$, $i = \overline{1,k}$.

One can readily see that $k \leqslant 2^{k-1} \leqslant \dfrac{\prod_{i=1}^{k} m_k}{\max\limits_{i=\overline{1,k}} m_i}$. Then, $\sum\limits_{i=1}^{k} m_i \leqslant k \max\limits_{i=\overline{1,k}} m_i \leqslant 2^{k-1} \max\limits_{i=\overline{1,k}} m_i \leqslant \dfrac{\prod_{i=1}^{k} m_k}{\max\limits_{i=\overline{1,k}} m_i} \max\limits_{i=\overline{1,k}} m_i = \prod\limits_{i=1}^{k} m_k$, which proves Lemma 1.

Proof of Lemma 2. Let $f_1, f_2, \ldots, f_k$ be a decomposition of the set $N(s)$. Similar to the proof of Lemma 1, we denote for convenience $|E_{f_i}| = m_i \geqslant 2$, $i = \overline{1,k}$, and $|E_{N(s)}| = \prod_{i=1}^{k} m_i$.

One can readily see that $k-1 \leqslant 2^{k-2} \leqslant \frac{\prod_{i=1}^{k} m_k}{\max_{i=\overline{1,k-1}} m_i m_{i+1}}$. Then, $\sum_{i=1}^{k-1} m_i m_{i+1} \leqslant (k-1) \max_{i=\overline{1,k-1}} m_i m_i \leqslant$ $2^{k-2} \max_{i=\overline{1,k-1}} m_i m_i \leqslant \frac{\prod_{i=1}^{k} m_k}{\max_{i=\overline{1,k-1}} m_i m_i} \max_{i=\overline{1,k-1}} m_i m_i = \prod_{i=1}^{k} m_k$, which proves Lemma 2.

Proof of Theorem 2. Let $f_1, f_2, \ldots, f_k$ be a decomposition of the set $N(s)$. We consider an arbitrary problem $w \in W^D(f_1, f_2, \ldots, f_k)$. According to Definition 6, the problem $w = w(s; M; n_1, n_2, \ldots, n_s; \{a_{F_{\overline{f}}}\}, \{b_{F_{\overline{f}}}\}, f \in M; \{c_{F_{N(s)}}\})$, where M and $\{c_{F_{N(s)}}\}$ are the $f_1, f_2, \ldots, f_k$-decompositions. Without loss of generality we assume that $M = \{\overline{f_i} | i = \overline{1,k}\} \cup \{\overline{f_i \cup f_{i+1}} | i = \overline{1, k-1}\}$; otherwise, we add the missing two-sided inequalities and take zero as the lower bound and a sufficiently great value such as $\sum_{F_{\overline{f'}} \in E_{\overline{f'}}} b_{F_{\overline{f'}}}$, where f' is an arbitrary element of M, as the upper bound. Now we present the procedure for constructing the problem $v = v(G; l_{ij}, u_{ij}, e_{ij}, (i,j) \in A_G) \in W_{Graph}$ which corresponds to the problem w. We construct an oriented graph G with the sets of vertices, $V_G = \{v'_{F_{f_i}}, v''_{F_{f_i}} | F_{f_i} \in E_{f_i},\ i = \overline{1,k}\} \cup \{s, t\}$, and arcs, $A_G = A_1 \cup A_2 \cup A_3$, where

—$A_1 = \{(v'_{F_{f_i}}, v''_{F_{f_i}}) |\, F_{f_i} \in E_{f_i},\ i = \overline{1,k}\}$;

—$A_2 = \{(v''_{F_{f_i}}, v'_{F_{f_{i+1}}}) |\, F_{f_i} \in E_{f_i}, F_{f_{i+1}} \in E_{f_{i+1}}, i = \overline{1,\ k-1}\}$;

—$A_3 = \{(s, v'_{F_{f_1}}) |\, F_{f_1} \in E_{f_1}\} \cup \{(v'_{F_{f_k}}, t) |\, F_{f_k} \in E_{f_k}\} \cup \{(t, s)\}$,

and define the function α as follows:

—the arc $(v'_{F_{f_i}}, v''_{F_{f_i}})$ is assigned to each constraint $d(w, \overline{f_i}, F_{f_i})$; therefore, in the problem v the lower and upper bounds of this arc are, respectively, $a_{F_{f_i}}$ and $b_{F_{f_i}}$, $F_{f_i} \in E_{f_i}$, $i = \overline{1,k}$;

—the arc $(v''_{(F_{f_i \cup f_{i+1}})_{f_i}}, v'_{(F_{f_i \cup f_{i+1}})_{f_{i+1}}})$ is assigned to each constraint $d(w, \overline{f_i \cup f_{i+1}}, F_{f_i \cup f_{i+1}})$; therefore, in the problem v the lower and upper bounds of this arc are, respectively, $a_{F_{f_i \cup f_{i+1}}}$ and $b_{F_{f_i \cup f_{i+1}}}$, $F_{f_i \cup f_{i+1}} \in E_{f_i \cup f_{i+1}}$, $i = \overline{1,\ k-1}$.

Since $w \in W^D(f_1, f_2, \ldots, f_k)$, according to Definitions 5 and 6 there are multi-index matrices $\{d_{F_{f_i}}\}$, $f \in B$ such that $c_{F_{N(s)}} = \sum_{f \in B} d_{(F_{N(s)})_f}$, where $B = \{f_i |\, i = \overline{1,k}\} \cup \{f_i \cup f_{i+1} |\, i = \overline{1,\ k-1}\}$. Then, the costs of arcs in the problem v are as follows:

—$e_{v'_{F_{f_i}} v''_{F_{f_i}}} = d_{F_{f_i}}$, $F_{f_i} \in E_{f_i}$, $i = \overline{1,k}$;

—$e_{v''_{F_{f_i}} v'_{F_{f_{i+1}}}} = d_{F_{f_i} F_{f_{i+1}}}$, $F_{f_i} \in E_{f_i}$, $F_{f_{i+1}} \in E_{f_{i+1}}$, $i = \overline{1,\ k-1}$;

—$e_{ij} = 0$, $(i,j) \in A_3$.

By condition, $f_1, f_2, \ldots, f_k$ is the decomposition of the set $N(s)$. Therefore, each element is representable unambiguously as $F_{N(s)} = F_{f_1} F_{f_2} \ldots F_{f_k}$. We denote $C_{F_{N(s)}} = (s, v'_{F_{f_1}}, v''_{F_{f_1}}, v'_{F_{f_2}}, v''_{F_{f_2}}, \ldots, v'_{F_{f_k}}, v''_{F_{f_k}}, t, s)$. By construction, $C(G) = \{C_{F_{N(s)}} |\, F_{N(s)} \in E_{N(s)}\}$. Then, the function β is defined as follows: a simple cycle $C_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$, is assigned to each variable $x_{F_{N(s)}}$.

We prove that the constructed problem v is consistent if and only if the original problem w is consistent as well. Indeed, let $x_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$, be a permissible solution of the constraint system of the problem w. We determine $y_{C_{F_{N(s)}}} = x_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$. Now, let $z_{ij} = \sum\limits_{C \in C(G)|(i,j) \in C} y_C$, $(i,j) \in A_G$. The set z_{ij}, $(i,j) \in A_G$, is the circulation in the graph G and by construction

$$—z_{v'_{F_{f_i}} v''_{F_{f_i}}} = \sum_{F_{\overline{f_i}} \in E_{\overline{f_i}}} x_{F_{\overline{f_i}} F_{f_i}}, \quad F_{f_i} \in E_{f_i}, \quad i = \overline{1, k};$$

$$—z_{v''_{(F_{f_i \cup f_{i+1}})_{f_i}} v'_{(F_{f_i \cup f_{i+1}})_{f_{i+1}}}} = \sum_{F_{\overline{f_i \cup f_{i+1}}} \in E_{\overline{f_i \cup f_{i+1}}}} x_{F_{\overline{f_i \cup f_{i+1}}} F_{f_i \cup f_{i+1}}}, \; F_{f_i \cup f_{i+1}} \in E_{f_i \cup f_{i+1}}, \; i = \overline{1,\, k-1}.$$

Then, the set z_{ij}, $(i,j) \in A$, according to the introduced function α, is the permissible circulation of the problem v.

Now, let z_{ij}, $(i,j) \in A_G$, be the permissible circulation of the problem v. According to Assertion 1, for the given circulation there exists its cyclic decomposition y_C, $C \in C(G)$. By construction,

$$—a_{F_{f_i}} \leqslant \sum_{F_{\overline{f_i}} \in E_{\overline{f_i}}} y_{C_{F_{\overline{f_i}} F_{f_i}}} = z_{v'_{F_{f_i}} v''_{F_{f_i}}} \leqslant b_{F_{f_i}}, \quad F_{f_i} \in E_{f_i}, \quad i = \overline{1, k};$$

$$—a_{F_{f_i \cup f_{i+1}}} \leqslant \sum_{F_{\overline{f_i \cup f_{i+1}}} \in E_{\overline{f_i \cup f_{i+1}}}} y_{C_{F_{\overline{f_i \cup f_{i+1}}} F_{f_i \cup f_{i+1}}}} = z_{v''_{(F_{f_i \cup f_{i+1}})_{f_i}} v'_{(F_{f_i \cup f_{i+1}})_{f_{i+1}}}} \leqslant b_{F_{f_i \cup f_{i+1}}}, \; F_{f_i \cup f_{i+1}} \in E_{f_i \cup f_{i+1}}, \quad i = \overline{1,\, k-1}.$$

According to the introduced function β, we construct the following set of values of variables: $x_{F_{N(s)}} = y_{C_{F_{N(s)}}}$, $F_{N(s)} \in E_{N(s)}$, which is a permissible solution of the problem w.

Let $z_{ij}, (i,j) \in A_G$, be a minimum-cost circulation in the problem v and $y_C, C \in C(G)$, its cyclic decomposition. With the use of the function β we construct the following set of values of the variables $x_{F_{N(s)}} = y_{C_{F_{N(s)}}}$, $F_{N(s)} \in E_{N(s)}$, which was shown above to be a permissible solution of the problem w. Now we prove by contradiction that the constructed set is the optimal solution of the problem w. Indeed, let this be not the case, and $x'_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$, be the optimal solution of the problem w. Then, by the assumption $\sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x_{F_{N(s)}} > \sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x'_{F_{N(s)}}$. By construction, $\sum\limits_{(i,j) \in A_G} e_{ij} z_{ij} = \sum\limits_{(i,j) \in A_G} e_{ij} \sum\limits_{C \in C(G)|(i,j) \in C} y_C = \sum\limits_{C \in C(G)} y_C \sum\limits_{(i,j) \in C} e_{ij} = \sum\limits_{F_{N(s)} \in E_{N(s)}} y_{C_{F_{N(s)}}} \sum\limits_{f \in B} d_{(F_{N(s)})_f} = \sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x_{F_{N(s)}}$. We determine $y'_{C_{F_{N(s)}}} = x'_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$, and $z'_{ij} = \sum\limits_{C \in C(G)|(i,j) \in C} y'_C$, $(i,j) \in A_G$. As was already shown, the so-constructed set z'_{ij}, $(i,j) \in A_G$, is a permissible circulation of the problem v. At that, $\sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x'_{F_{N(s)}} = \sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} y'_{F_{N(s)}} = \sum\limits_{F_{N(s)} \in E_{N(s)}} y'_{C_{F_{N(s)}}} \sum\limits_{f \in B} d_{(F_{N(s)})_f} = \sum\limits_{C \in C(G)} y'_C \sum\limits_{(i,j) \in C} e_{ij} = \sum\limits_{(i,j) \in A_G} e_{ij} \sum\limits_{C \in C(G)|(i,j) \in C} y'_C = \sum\limits_{(i,j) \in A_G} e_{ij} z'_{ij}$. Then, $\sum\limits_{(i,j) \in A_G} e_{ij} z_{ij} = \sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x_{F_{N(s)}} > \sum\limits_{F_{N(s)} \in E_{N(s)}} c_{F_{N(s)}} x'_{F_{N(s)}} = \sum\limits_{(i,j) \in A_G} e_{ij} z'_{ij}$, that is, we get a contradiction. Therefore, the above assumption is incorrect and the constructed set $x_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$, is the optimal solution of the problem w.

We analyze complexity of the computing procedures related with construction of the problem v and solution of the problem w from the solution of the problem v. We notice that the original problem w has $|E_{N(s)}|$ variables. By construction, in the problem v the graph G has $|V| = 2 + 2\sum\limits_{i=1}^{k} |E_{f_i}|$ vertices and $|A| = 1 + |E_{f_i}| + |E_{f_k}| + \sum\limits_{i=1}^{k-1} |E_{f_i}||E_{f_{i+1}}|$ arcs. Lemmas 1 and 2 enable us to establish the following estimates $|V| = O(|E_{N(s)}|)$, $|A| = O(|E_{N(s)}|)$. Whence it follows that construction of the problem v creates linear computer burden vs. to the dimension of the problem w. Solution of the problem w from the solution of the problem v entails construction of the cyclic decomposition of the circulation on the graph G. By Assertion 2, construction of the cyclic decomposition requires $O(|V||A|) = O(|E_{N(s)}|^2)$ computing operations. Whence it follows that the class $W(f_1, f_2, \dots, f_k)$ is $L|L-equal||E_{N(s)}|^2-cycle$ reducible to the class W_{Graph}, which proves Theorem 2.

Proof of Theorem 3. Let $f_1, f_2, \dots, f_k$ be the decomposition of the set $N(s)$ and $k \geqslant 3$. We consider the set $M = \{\overline{f_1}, \overline{f_2}, \overline{f_3}\}$. According to Definition 4, M is the $f_1, f_2, \dots, f_k$-decomposition set. We prove that the equality P = NP follows from existence of the polynomial ε-approximate algorithm of solution of problems of the class $W_Z(M)$, where ε is an arbitrary nonnegative constant.

We consider the well-known NP-complete problem of the tri-dimensional combination [19] which can be formalized as follows. Let $U \subseteq \{1, 2, \dots, m\}^3$, where $m \in N$. Then, the problem lies in verifying for consistence the following integer system of linear inequalities:

$$\sum_{i=1}^{m}\sum_{j=1}^{m} y_{ijk} = 1, \quad k = \overline{1, m};$$

$$\sum_{i=1}^{m}\sum_{k=1}^{m} y_{ijk} = 1, \quad j = \overline{1, m};$$

$$\sum_{j=1}^{m}\sum_{k=1}^{m} y_{ijk} = 1, \quad i = \overline{1, m};$$

$$y_{ijk} \in \{0, 1\}, \quad (i, j, k) \in U;$$

$$y_{ijk} = 0, \quad (i, j, k) \in \{1, 2, \dots, m\}^3 \backslash U.$$

We take a problem $w_Z \in W_Z(M)$ satisfying the conditions $|E_{f_1}| = |E_{f_2}| = |E_{f_3}| = m$ and numerate the elements of the set E_{f_l}, $l = \overline{1, 3}$, as follows: $E_{f_l} = \{F_{f_l,1}, F_{f_l,2}, \dots, F_{f_l,m}\}$, $l = \overline{1, 3}$. Since f_1, f_2, f_3, f' is a decomposition of the set $N(s)$, where $f' = N(s) \backslash (f_1 \cup f_2 \cup f_3)$, any element of $F_{N(s)} \in E_{N(s)}$ is representable as $F_{N(s)} = F_{f_1}F_{f_2}F_{f_3}F_{f'}$. The costs $c_{F_{N(s)}}$, $F_{N(s)} \in E_{N(s)}$, are defined as follows. Let $c_{F_{N(s)}} = 0$ if $F_{N(s)} = F_{f_1,i}F_{f_2,j}F_{f_3,k}F_{f'}$, where $(i, j, k) \in U$ and $F_{f'} = (1, 1, \dots, 1)$, and $c_{F_{N(s)}} = 1$, otherwise. Finally, we determine the values of the free coefficients of the two-sided inequalities in the problem w_Z. Let $a_{F_{f_l}} = b_{F_{f_l}} = 1$, where $F_{f_l} \in E_{f_l}$, $l = \overline{1, 3}$.

We denote by $OPT(w_Z)$ the optimal value of the criterion of the selected problem w_Z. We can readily see that $OPT(w_Z) = 0$ if and only if the constraint system of the problem of tri-dimensional combination is consistent. At that, if $OPT(w_Z) \neq 0$, then $OPT(w_Z) \geqslant 1$.

Let there be a polynomial ε-approximate algorithm H for solution of problems of the class $W_Z(M)$, where $\varepsilon \geqslant 0$. We denote by $H(w_z)$ the permissible solution established by the algorithm H; the value of the criterion of the problem w_Z on the solution $H(w_z)$ is denoted by $c(H(w_z))$. If $c(H(w_z)) = 0$, then $OPT(w_Z) = 0$ and the constraint system of the problem of tri-dimensional combination is consistent. If $c(H(w_z)) \neq 0$, then $c(H(w_z)) \geqslant 1$. Yet, since H is polynomial and ε-approximate the condition $1 \leqslant c(H(w_z)) \leqslant OPT(w_Z)(1 + \varepsilon)$ is satisfied, whence it follows that $OPT(w_Z) \geqslant 1/(1 + \varepsilon) > 0$. Since $OPT(w_Z) \neq 0$, the constraint system of the problem of tri-dimensional combination is inconsistent.

Consequently, one may suggest the following polynomial algorithm to solve the problem of tri-dimensional combination. According to the aforementioned scheme, construct the corresponding problem $w_Z \in W_Z(M)$ for the selected problem of tri-dimensional combination. Apply then the polynomial ε-approximate algorithm H to solve the problem w_Z. If $c(H(w_z)) = 0$, then the constraint system of the problem of tri-dimensional combination is consistent; otherwise, inconsistent. Yet, since the problem of tri-dimensional combination is NP-complete, it follows from the existence of the polynomial algorithm for its solution that P = NP [19], which proves Theorem 3.

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