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LOGISTICS

Multiindex Optimal Production Planning Problems

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Abstract—We consider production planning problems formalized as optimization problems with a multi-index constraint system of the transport type. These problems arise, for instance, upon constructing a portfolio of orders, master scheduling, etc. We consider computational schemes of solving this problem for different kinds of optimization functions.

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1. INTRODUCTION

At present, planning processes automation for industrial systems has become increasingly relevant due to the intensification of production. The following classes of problems are usually considered: the problem of constructing a portfolio of orders, master scheduling problems, and calendar planning problems. Calendar planning problems are considered in scheduling theory and are, in essence, distribution problems for constrained resources over time. A lot of attention has been devoted to these problems in scientific literature [1–5]. Order portfolio design and master scheduling are less known. The order portfolio design problem is to determine which orders a factory is going to fulfill over the planning period. These portfolios are usually designed for an upcoming year. The master scheduling problem is to distribute the production schedule of a company, found after solving the order portfolio problem, into calendar subperiods (months). These problems are not considered in the general setting, taking into account all connections and dependencies, as in calendar planning problems, but usually with a certain degree of idealization. Instead of specific activities with their durations one considers volumetrics (labor hours, roubles, unit tons) related to a collection of activities that comprise an order.

We will consider the order portfolio and master scheduling problems as distribution problems for limited resources in multi-index hierarchical systems of the transport type [6–10]. The resource allocation problem has been a subject of many studies (see, e.g., [11–15]). An important difference of our work is that we formalize problems of this class as optimization problems with a multi-index system of constraints of the transport type. In this work, we give the settings of problems arising upon designing an order portfolio and master scheduling. We consider computational schemes for solving the corresponding multi-index optimization problems.

2. PROBLEM SETTINGS

2.1. Order Portfolio Design Problem

We have to construct a portfolio of orders for a factory that would satisfy certain predefined general bounds on the factory's productivity and the orders' labor intensity. Let I be the set of departments of the factory, J be the set of orders, K be the set of items. By A_i we denote the "productivity" of a subdivision i, i.e., the total amount of work that department i can perform over the planning period; B_k denotes the amount of work planned to be done by the factory over the planning period with respect to item k; C_{ik} denotes the amount of work planned to be done by the ith department for the kth order; D_{jk} is the amount of work planned to be done for the kth item of the jth order; e_{ijk} is the profit that the factory plans to earn for making a single unit of the amount of work done in department i on item k of order j, $i \in I$, $j \in J$, $k \in K$.

The order portfolio design problem is to determine the values of x_{ijk}, i.e., the amount of work that will be done in department i on item k of order j, $i \in I$, $j \in J$, $k \in K$. For these values, the following constraints hold:

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} \leqslant A_i, \quad i \in I, \tag{1}$$

$$\sum_{i\in I}\sum_{j\in J} x_{ijk} \geqslant B_k, \quad k \in K, \tag{2}$$

$$\sum_{j\in J} x_{ijk} \leqslant C_{ik}, \quad i \in I, \quad k \in K, \tag{3}$$

$$\sum_{i\in I} x_{ijk} \geqslant D_{jk}, \quad j \in J, \quad k \in K, \tag{4}$$

$$x_{ijk} \geqslant 0, \quad i \in I, \quad j \in J, \quad k \in K, \tag{5}$$

the following criterion:

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} e_{ijk}x_{ijk} \to \max, \tag{6}$$

which characterizes the total profit made by the company, is to be maximized. The constraints we have introduced mean:

(1) that the amount of work done for all orders and items in the ith department must not exceed the "productivity" of this department;

(2) that the amount of work planned for the kth item must be done in the company's departments over the planning period;

(3) that the planned amount of work must not exceed the amount of work that will be done by the ith department for the kth item;

(4) that the amount of work planned for the kth item of the jth order must be done in the company's departments during the planning period;

(5) means natural conditions on the variables.

In order for the system of constraints (1)–(5) to be feasible, the following conditions must hold: $\sum_{i\in I} C_{ik} \geqslant B_k$, $\sum_{i\in I} C_{ik} \geqslant \sum_{j\in J} D_{jk}$, $k \in K$. Since the company is not interested in making extra product, the problem setting implies that the original values are related by the following natural conditions: $\sum_{i\in I} C_{ik} = B_k$, $\sum_{i\in I} C_{ik} = \sum_{j\in J} D_{jk}$, $k \in K$, for which constraints (2), (3), (4) should hold as equalities. In this setting, the system of constraints (1)–(5) may be infeasible due to the failure of (1): the total productivity of all departments may be insufficient to fulfill the orders which the company would like to include in the production plan. If the departments' productivities can be increased, the following problem setting becomes natural.

We denote by q_i the costs of increasing the productivity of the ith department by one, by y_i the value by which the ith department's productivity will be increased, and by Q_i the value by which the ith department productivity can be increased, $i \in I$.

Then the order portfolio design problem is transformed into a problem with the original problem's constraints (2)–(5), new constraints

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} \leqslant A_i + y_i, \quad i \in I; \tag{7}$$

$$0 \leqslant y_i \leqslant Q_i, \quad i \in I, \tag{8}$$

and criterion

$$\sum_{i\in I} q_i y_i \to \min \tag{9}$$

that characterizes the total costs of increasing productivity across the company. The newly introduced constraints have the following meaning:

(7) means that the amount of work done across all orders and items in the ith department must not exceed the department's productivity, taking into account its possible increase;

(8) means that the productivity increase for the ith department must not exceed the maximal possible value.

2.2. The Master Scheduling Problem

The master scheduling problem is to distribute the production schedule, found at the order portfolio design stage, over calendar subperiods. In addition to the sets I, J, K that we have already introduced for the previous problem, we introduce the set $T = \{1, 2, \ldots, T_0\}$ of numbers of planning cycles and values t_j^-, t_j^+ denoting the start tick number (start time) and the end tick number (schedule time) of fulfilling order j, $t_j^-, t_j^+ \in T$, $j \in J$. The initial parameters of the master planning problem will include, A_{it}, B_{ijt}, C_{ikt}, D_{ijk}, E_{ijkt}, $i \in I$, $j \in J$, $k \in K$, $t \in T$, i.e., respectively, the factory's departments capacities per planning tick, the amounts of work that must be done in relation to the orders in the departments per planning tick, the amounts of work in orders and items that must be done in the company departments, the amounts of work that must be done in the company departments in relation to orders and items per planning tick. There is a natural connection between the introduced parameters and the solution of the order portfolio design problem. Let x^0_{ijk}, $i \in I$, $j \in J$, $k \in K$, be a solution of the order portfolio design problem. Then $\sum\limits_{t\in T} B_{ijt} = \sum\limits_{k\in K} x^0_{ijk}$, $\sum\limits_{t\in T} C_{ikt} = \sum\limits_{j\in J} x^0_{ijk}$, $D_{ijk} = x^0_{ijk}$, $i \in I$, $j \in J$, $k \in K$.

The problem of distributing a factory's schedule into calendar subperiods is to determine the values of z_{ijkt}, i.e., the amounts of work that will be done in department i for item k of order j during tick t, $i \in I$, $j \in J$, $k \in K$, $t \in T$, for which the following constraints hold:

$$\sum_{j\in J}\sum_{k\in K} z_{ijkt} \leqslant A_{it}, \quad i \in I, \quad t \in T, \tag{10}$$

$$\sum_{k\in K} z_{ijkt} \geqslant B_{ijt}, \quad i \in I, \quad j \in J, \quad t \in T, \tag{11}$$

$$\sum_{j\in J} z_{ijkt} \geqslant C_{ikt}, \quad i \in I, \quad k \in K, \quad t \in T, \tag{12}$$

$$\sum_{t\in T} z_{ijkt} \geqslant D_{ijk}, \quad i \in I, \quad j \in J, \quad k \in K, \tag{13}$$

$$z_{ijkt} \leqslant E_{ijkt}, \quad i \in I, \quad j \in J, \quad k \in K, \quad t \in T, \tag{14}$$

$$z_{ijkt} = 0, \quad t \in T \setminus \left\{t_j^-, t_j^- + 1, \ldots, t_j^+\right\}, \quad i \in I, \quad j \in J, \quad k \in K, \tag{15}$$

$$z_{ijkt} \geqslant 0, \quad i \in I, \quad j \in J, \quad k \in K, \quad t \in T. \tag{16}$$

The introduced constraints mean:

(10) the amount of work done by all orders and items in the ith department must not exceed the "capacity" of this department during tick t;

(11) the amount of work planned for the ith department during tick t on order j must be done;

(12) the amount of work planned for the ith department during tick t on item k must be done;

(13) the amount of work planned for the ith department for item k of order j must be done;

(14) constraints on excess production;

(15) constraints for keeping up with initial and schedule times for the orders;

(16) natural conditions on the variables.

Among all admissible plans we are looking for the plans that satisfy the conditions of efficient operation for a production system. For a master planning problem, the operation efficiency depends on several factors. As experience in solving these problems has shown [7, 9], it is hard to formalize these factors as continuous functions for the user. The user is more likely to be able to estimate the parameters of the desired plan by setting the constraints on the values of deviations inside which these values are "perfect," "very good," "good," "satisfactory," etc. Then the formalized criteria of the master planning problem can be represented as piecewise constant functions that break the set of deviation values up into deviation "quality" regions for each criterion. These functions can be functions whose range is given by the set of nonnegative integers from 0 to p (0 meaning "perfect," 1—"very good" and so on).

Segments of possible values for some plan parameters are set, and they determine the operation efficiency for the production system. Plan parameters can be any parameters determined by constraints (10)–(16). Let the system operation efficiency be related to using the departments' capacities in planning ticks defined by the set $H \subseteq I \times T$. Then these segments will be $[0, A_{it}]$, $(i,t) \in H$. Further, for each of the plan parameters related to segment $[0, A_{it}]$ we define a collection of $p+1$ nested segments $S_{it}^{(l)}$, $l = \overline{0,p}$, such that $S_{it}^{(l)} \subseteq S_{it}^{(l+1)}$, $l = \overline{0,p-1}$, $S_{it}^{(p)} = [0, A_{it}]$, $(i,t) \in H$. With each of these parameters, we associate a preference function $\chi_{it}(w, S_{it}^{(0)}, S_{it}^{(1)}, \ldots, S_{it}^{(p)})$ that takes value r if $w \in S_{it}^{(r)}$ and $w \notin S_{it}^{(r-1)}$, $r = \overline{0,p}$, $(i,t) \in H$. Then we will consider the problem of finding an admissible plan z_{ijkt}, $i \in I$, $j \in J$, $k \in K$, $t \in T$, satisfying the system of constraints (10)–(16) and minimizing the preference functions for selected parameters:

$$\chi_{it}\left(\sum_{k\in K}\sum_{j\in J} z_{ijkt}, S_{it}^{(0)}, S_{it}^{(1)}, \ldots, S_{it}^{(p)}\right) \to \min, \quad (i,t) \in H. \tag{17}$$

The problem (10)–(17) is a multicriterial optimization problem. Assuming that the chosen plan parameters are sorted in the order of their importance with relation to the factory operation efficiency, as a compromise scheme we consider the lexicographic ordering of individual optimality criteria.

To formalize the compromise scheme for the considered multicriterial problem, we introduce several auxiliary values. Let the elements of H be ordered based on the priorities of the chosen parameters, and $H = \{(i_1, t_1), (i_2, t_2), \ldots, (i_{|H|}, t_{|H|})\}$. Let $a, b \in N$. We introduce the set $V_{a,b} = \{(v_1, v_2, \ldots, v_b) | v_l = \overline{1,a}, l = \overline{1,b}\}$. Then by $V_{a,b}$ we will denote (similarly to [6, 9]) the set of vertices of an a-valued b-dimensional cube. Further, we introduce the $(p+1)$-valued $|H|$-dimensional cube on which we define an ordering Π. To each cube vertex $\vec{v} \in V_{(p+1),|H|}$ we associate a system $\Omega(\vec{v})$. The system $\Omega(\vec{v})$ contains constraints (10)–(16) independent of the cube vertex and constraints dependent on the vertex in the following way: if $v_l = s$ then we add $\sum\limits_{k\in K}\sum\limits_{j\in J} z_{i_l jkt_l} \in S_{i_l t_l}^{(s)}$, $l = \overline{1, |H|}$, to the system. On the set of cube vertices, we define a binary function $g(\vec{v})$ which is one if the system $\Omega(\vec{v})$ is feasible and zero otherwise. By $V = \{\vec{v} \in V_{(p+1),|H|} | g(\vec{v}) = 1\}$ we denote the set of vertices of the $(p+1)$-valued $|H|$-dimensional cube, for which the corresponding function g equals one.

As the ordering Π, we consider the lexicographic ordering: $\vec{v}^1 \Pi \vec{v}^2$ if and only if for some l, $l = \overline{1, |H|}$, $v_l^1 < v_l^2$ and, at the same time, $v_r^1 = v_l^2$ for $r = \overline{1, l-1}$.

We now consider the following problem of finding the optimal cube vertex $\vec{v}^0$:

$$\vec{v}^0 \in V, \tag{18}$$

$$\vec{v}^0 \Pi \vec{v}, \quad \vec{v} \in V. \tag{19}$$

The optimal vertex $\vec{v}^0$, which is a solution of (18), (19), defines an optimal solution of the multicriterial problem (10)–(17) for a lexicographical compromise scheme.

Remark. For the problem setting (18), (19), the production system operation efficiency parameters may include conditions defined not by a single group of constraints but by several groups. Moreover, we can account for all subsets of the constraints in selected groups.

3. SOLUTION ALGORITHMS

Problems posed in Section 2 are related to studying optimization problems whose system of constraints is a multi-index system of linear inequalities of the transport type. To describe multiindex problems, we use the following formalization. Suppose that we are given a set of indices $N(s) = \{i_1, i_2, \ldots, i_s\}$ and a set $M \subseteq 2^{N(s)}$. Then by $W(M)$ we denote the multi-index linear programming problem of transport type with the set of indices $N(s)$ and the system of constraints that consists, for each $f \in M$, of constraints for subsums in which the summation is over all indices of the set f for fixed sets of index values from $N(s) \setminus f$. For convenience, we denote by $E_{N(s)}$ the set of all possible values of s-index sets of indices from the set $N(s)$; by $D(M)$ we denote the system of constraints for the problem $W(M)$.

In genera, only universal linear programming techniques can be used to solve the problem $W(M)$. However, the specifics of our problems (linear constraints of the transport type) allows us to introduce more efficient algorithms for a particular class of considered problems by reducing them to flow algorithms [8]. In what follows we give the results of reducing $W(M)$ to flow algorithms found in [8].

Definition 1. A set M, $M \subseteq 2^{N(s)}$, is called k-nesting if there exists a partition of set M into k subsets $M_i = \{f_1^{(i)}, f_2^{(i)}, \ldots, f_{m_i}^{(i)}\}$, $i = \overline{1, k}$, such that $f_j^{(i)} \subseteq f_{j+1}^{(i)}$, $j = \overline{1, m_i - 1}$, $i = \overline{1, k}$.

Theorem 1. *In order for the problem $W(M)$ to be reducible to a minimal flow problem, it suffices for the set M to be 2-nesting.*

Corollary 1. *If the set M is 2-nesting then problem $W(M)$ (system of inequalities $D(M)$) reduces to finding the minimal flow (admissible flow) in a network with $O(|E_{N(s)}|)$ vertices and $O(|E_{N(s)}|)$ edges.*

Further, using known flow algorithms (see [16, 17]), we can formulate the following result.

Corollary 2. *If the set M is 2-nesting then there exists an algorithm for the problem $W(M)$ (system of inequalities $D(M)$) that takes $O(|E_{N(s)}|^3 \log^2 |E_{N(s)}|)$ $(O(|E_{N(s)}|^2 \log |E_{N(s)}|))$ computational steps.*

Problem (1)–(6) corresponds to problem $W(M)$ for $s = 3$, $N(s) = \{i, j, k\}$, $M = \{\{j, k\}, \{i, j\}, \{j\}, \{i\}\}$, $E_{N(s)} = I \times J \times K$. For the set M, there exists a partition $M_1 = \{\{j, k\}, \{j\}\}$, $M_2 = \{\{i, j\}, \{i\}\}$, so the set M is 2-nesting. Therefore, by applying the proposed approach and using

Corollary 2, we can construct an algorithm for the order portfolio design problem (1)–(6) that requires $O(|I \times J \times K|^3 \log^2 |I \times J \times K|)$ computational steps and an algorithm for deciding the feasibility of system (1)–(5) that requires $O(|I \times J \times K|^2 \log |I \times J \times K|)$ computational steps.

Suppose that the system of constraints $D(M)$ is infeasible. A subset of two-sided inequalities in the system is known for which left and/or right borders are allowed to move. We call constraints with allowed changes in borders "desired," and other constraints we call "hard." There are known costs for violating desired constraints. The problem is to find unknown values that satisfy hard constraints and minimize the total cost of violating desired constraints. We denote this problem by $L(M)$.

According to Corollary 1, in case the set M is 2-nesting studying the system of constraints $D(M)$ reduces to finding an admissible flow in a network. Thus, problem $L(M)$ reduces studying an infeasible system $D(M)$ to studying an infeasible network model. In [18], it has been shown that the flow search problem in an infeasible network with n vertices and m edges reduces to finding a minimal flow in a network with $O(n + m)$ vertices and $O(m)$ edges. Then, using Corollary 1 and applying known minimal flow algorithms [16, 17], we can formulate the following results.

Corollary 3. *If the set M is 2-nesting then the problem $L(M)$ reduces to the minimal flow problem in a network with $O(|E_{N(s)}|)$ vertices and $O(|E_{N(s)}|)$ edges.*

Corollary 4. *If the set M is 2-nesting then there exists an algorithm for the problem $L(M)$ that requires $O(|E_{N(s)}|^3 \log^2 |E_{N(s)}|)$ computational steps.*

The problem (2)–(5), (7)–(9) corresponds to the problem $L(M)$ with $s = 3$, $N(s) = \{i, j, k\}$, $M = \{\{j, k\}, \{i, j\}, \{j\}, \{i\}\}$, $E_{N(s)} = I \times J \times K$, and the set M, as we have already shown, is 2-nesting. Therefore, using the proposed approach and applying Corollary 4, we can construct an algorithm for the order portfolio design problem (2)–(5), (7)–(9) that requires $O(|I \times J \times K|^3 \log^2 |I \times J \times K|)$ computational steps.

In [9], an algorithm for finding the optimal vertex in a multidimensional multivalued cube was devised. It was proven that finding an optimal vertex for a $(p + 1)$-valued $|H|$-dimensional cube reduces to a feasibility check for about $|H| \log_2(p + 1)$ systems of the form $\Omega(\vec{v})$. When solving problem (18), (19) that arises in master planning, the corresponding system $\Omega(\vec{v})$ is a system of the form $D(M)$, where $s = 4$, $N(s) = \{i, j, k, t\}$, $M = \{\{j, k\}, \{k\}, \{j\}, \{t\}, \emptyset\}$. The set M is not 2-nesting in this case. To check feasibility in the corresponding system $D(M)$, we can use the Agmon–Motzkin orthogonal projection method [19, 20] suitably modified in [9]. The Agmon–Motzkin orthogonal projection method is an iterative algorithm. On each step, a constraint violated by the current solution is determined. Then the current solution is projected on the hyperplane related to the violated constraint, and we proceed to the next step. In [19, 20], it is shown that this method converges.

In a special case when the system of constraints for a master planning problem is given by conditions (10), (11), (13)–(16), to solve the problem (18), (19) the corresponding system $\Omega(\vec{v})$ is a system of the kind $D(M)$, where $s = 4$, $N(s) = \{i, j, k, t\}$, $M = \{\{j, k\}, \{k\}, \{t\}, \emptyset\}$, $E_{N(s)} = I \times J \times K \times T$. For the set M there exists a partition $M_1 = \{\{j, k\}, \{k\}, \emptyset\}$, $M_2 = \{\{t\}\}$, therefore, M is 2-nesting. Then, by Corollary 2, we can construct a feasibility checking algorithm for the corresponding system $\Omega(\vec{v})$ that requires $O(|I \times J \times K \times T|^2 \log |I \times J \times K \times T|)$ computational steps. Therefore, the proposed algorithm for finding an optimal cube vertex will require $O(|I \times J \times K \times T|^2 \log |I \times J \times K \times T||H| \log p)$ computational steps.

4. CONCLUSION

The proposed approach to studying optimal planning problems for production systems allows to reduce a wide class of problems important in practice to well-developed efficient flow algorithms.

The software created on the basis of the results of this work was tested in solving master planning problems for factories with piece and small-batch production (Instrumental production FGUP "Yu.E. Sedakov FNPC NIIIS," number of orders—up to 25 000, number of departments—up to 100, planning horizon—1 year, planning tick—a month, solution time—up to 5 minutes [7]) and oil refineries (Surgut plant of condensate stabilization JSC "Surgutgazprom," number of orders—up to 100, number of departments—up to 5, number of ticks—up to 30, solution time—up to 10 minutes, [10]).

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