ISSN 0005-1179, Automation and Remote Control, 2006, Vol. 67, No. 6, pp. 1007–1016. © Pleiades Publishing, Inc., 2006.
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Multiindex Resource Distributions for Hierarchical Systems

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Received April 27, 2005

Abstract—Resource distribution in hierarchical systems is formulated as a multiindex linear programming problem under transport-type constraints. Conditions under which this problem is reduced to the determination of the minimal-cost circulation in a transport network are stated.

PACS number: 89.20.Ff

DOI: 10.1134/S0005117906060130

1. INTRODUCTION

Many applied problems are concerned with the distribution of limited resources in hierarchical systems [1–7], for example, balanced load distribution in a heterogeneous network [2], job distribution to parallel computers [3], resource distribution during negotiations [4], etc. A central point here is their formalization in the form of many-criteria multiindex problems with transport-type linear constraints [5–7]. They are described as follows. A system is assumed to contain three types of elements: source, transfer, and consumer elements. These elements and their relations obey resource constraints defining resource volumes that may circulate in the system. There are "controllable" elements, which determine conditions for the "effective" operation of the system. Every controllable element defines binary relations on a suitable admissible resource distribution interval. These relations are defined by preference functions defined for controllable elements. In its general formulation, the problem of resource distribution in a hierarchical system consists in determining an admissible resource distribution variant, in which preference functions take extremal values for controllable elements. Such problems can be formally represented by many-criteria problems under linear constraints and criteria, whose type depends on the type of preference functions. In [6], such functions are taken to be piecewise-linear and quadratic functions. If the operation of a system depends on economic indexes, such as total costs, income or profit, preference functions are taken to be linear functions. Assuming that the resource distribution in a system satisfies additivity and proportionality conditions, we can define compromise by a linear convolution of optimality criteria defined for controllable elements. Then resource distribution in a hierarchical system can be formulated as a multiindex linear programming problem. Examples of such problems are transportation with intermediate stations [8], load distribution to data transmission channels by Internet Service Providers [5], and production scheduling [7].

1.1. A Transport Problem with Intermediate Stations

There are production, intermediate, and consumption points for a homogeneous product. The maximum possible volumes of production at every production point and consumption at every consumption point, constraints on transport of the product from every production point to every intermediate point, and constraints on the volume of transport of the product from every intermediate point to every consumption point are given. Our problem now is to find a transport plan to minimize transport costs if the costs of transport of the product from production points to consumption points via intermediate points are known.

We use the following notation: I is a set of production points, J is a set of intermediate points, K is a set of consumption points, A_i is the maximal volume of production at point i, B_k is the volume of product to be transported to the kth consumption point, D_{jk} is the maximal volume of product that can be transferred from the jth intermediate point to the kth consumption point, E_{ij} is the maximal volume of the product that can be delivered from the ith production point to the jth intermediate point, and c_{ijk} is the cost of delivery per unit product from the ith production point via the jth intermediate point to the kth consumption point, $i \in I$, $j \in J$, $k \in K$. Then our problem is to find the product volume x_{ijk} that can be transported from the production point i via the jth intermediate point to the kth consumption point such that the constraints

$$\sum_{j\in J}\sum_{k\in K} x_{ijk} \le A_i, \quad i \in I,$$
$$\sum_{i\in I}\sum_{j\in J} x_{ijk} \ge B_k, \quad k \in K,$$
$$\sum_{i\in I} x_{ijk} \le D_{jk}, \quad j \in J, \quad k \in K,$$
$$\sum_{k\in K} x_{ijk} \le E_{ij}, \quad i \in I, \quad j \in J,$$
$$x_{ijk} \ge 0, \quad i \in I, \quad j \in J, \quad k \in K,$$

are satisfied and criterion

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ijk}x_{ijk}$$

characterizing the total transport costs is minimal.

1.2. Distribution of Traffic Load to Data Transfer Channels by an Internet Service Provider

The problem here is to distribute limited capacities of data transmission channels between different network nodes of city providers. Needs of network users for a particular amount of information are known. The possibilities of providers for providing channels of a particular capacity to different network nodes are known. The preferences of users and providers for transferring a particular amount of information to another user or node are known. Conditions for effective distribution of channels (depending on their capacities) are given. The network structure and information distribution in general are of diverse. We study the problem under the following constraints:

• information is transmitted from the center to users via nodes through communication channels,

• every network node or user is served by one or several nodes, and

• the amount of information distributed to nodes and users is bounded above (constraints on provider's capabilities) and below (users' minimal requirements for information).

Channel capacities must be distributed most effectively with regard for users' requirements and preferences and capabilities of providers.

We use the following notation: P is a set of providers, R is a set of communication nodes, U is a set of users, A_i^- and A_i^+ are the lower and upper bounds of the total power of the data transmission channel of provider i, B_j^- and B_j^+ are the lower and upper bounds of the total power of the data transmission channel at node j, C_k^- and C_k^+ are the lower and upper bounds of the total power of the data transmission channel that can be provided to user k, D^-_{ijk} and D^+_{ijk} are the lower and upper bounds of the power of the data transmission channel from provider i to user k via node j, h_i is the cost of communication per unit power provided by provider i, g_j is the cost of processing of information at transmission station j, and q_k is the income per unit information form user k, $i \in P$, $j \in R$, $k \in U$. Then, assuming that channel capacity distribution satisfies additivity and proportionality conditions, we can study the problem of maximization of the total profit, i.e., determine the communication channel capacity x_{ijk} supplied to user k via node j by provider i, $i \in P$, $j \in R$, $k \in U$, such that the constraints

$$A^-_i \le \sum_{j\in R}\sum_{k\in U} x_{ijk} \le A^+_i, \quad i \in P,$$
$$B^-_j \le \sum_{i\in P}\sum_{k\in U} x_{ijk} \le B^+_j, \quad j \in R,$$
$$C^-_k \le \sum_{i\in P}\sum_{j\in R} x_{ijk} \le C^+_k, \quad k \in U,$$
$$D^-_{ijk} \le x_{ijk} \le D^+_{ijk}, \quad i \in P, \quad j \in R, \quad k \in U,$$

are satisfied and the criterion

$$\sum_{i\in P}\sum_{j\in R}\sum_{k\in U} (q_k - h_i - g_j)x_{ijk}$$

defining the total profit of the system is maximal.

1.3. Production Scheduling

The problem here is to plan production with indexes in a given period satisfying certain given characteristics. We use the following notation: S is a set of production plants, Q is a set of orders, P is a set of products, T is a set of production cycles, A_t is the total work volume required to produce the products of all orders at all plants in a cycle t, B_j is the total work volume to be implemented in all cycles to produce all products at all plants to meet order j, E_{it} is the total work volume to be implemented to produce all products of all orders at plant i in cycle t, D_{ij} is the total work volume to be implemented in all cycles to produce all products at plant i for order j, F_{ijk} is the total work volume to be implemented in all cycles at plant i to produce product k for order j, and c_{jk} is the income of the system in the plan period from production of product k for order j, $i \in S$, $j \in Q$, $k \in P$, $t \in T$. Then the production planning consists in determining the work volume x_{ijkt} to be implemented in cycle t to produce product k for order j at plant i, $i \in S$, $j \in Q$, $k \in P$, $t \in T$, such that the constraints

$$\sum_{i\in S}\sum_{j\in Q}\sum_{k\in P} x_{ijkt} \ge A_t, \quad t \in T,$$
$$\sum_{i\in S}\sum_{k\in P}\sum_{t\in T} x_{ijkt} \ge B_j, \quad j \in Q,$$
$$\sum_{j\in Q}\sum_{k\in P} x_{ijkt} \ge E_{it}, \quad i \in S, \quad t \in T,$$
$$\sum_{k\in P}\sum_{t\in T} x_{ijkt} \ge D_{ij}, \quad i \in S, \quad j \in Q,$$
$$\sum_{t\in T} x_{ijkt} \le F_{ijk}, \quad i \in S, \quad j \in Q, \quad k \in P,$$
$$x_{ijkt} \ge 0, \quad i \in S, \quad j \in Q, \quad k \in P, \quad t \in T,$$

are satisfied and the criterion

$$\sum_{j\in Q}\sum_{k\in P} c_{jk} \sum_{i\in S}\sum_{t\in T} x_{ijkt}$$

defining the total income of the production system in the planned period takes the maximal value.

All these problems have certain common features, viz.,

• variable parameters of the model are multiindex and the number of indexes may be different, depending on the concrete problem,

• constraints of the model are defined by a system of linear algebraic transport-type inequalities, each of which is obtained by summing over certain indexes, and

• optimization criteria are defined by functions, whose arguments are also sums of values of variable parameters taken over certain indexes.

2. FORMULATION OF THE PROBLEM

The distribution of restricted resources in hierarchical systems with regard the common properties of all these problems is formulated through the formalization procedure developed in [8] as a multiindex linear programming transport problem.

Let $N(s) = \{1, 2, \dots, s\}$. Every number l is associated with a parameter j_l, called the index, which takes values in the set $J_l = \{1, 2, \dots, n_l\}$, $l \in N(s)$. Let $f = \{k_1, k_2, \dots, k_t\}$, $f \subseteq N(s)$. The set of values of indexes $F_f = (j_{k_1}, j_{k_2}, \dots, j_{k_t})$ is called the t-index, and set of all t-indexes is denoted by $E_f = J_{k_1} \times J_{k_2} \times \cdots \times J_{k_t}$. Every set F_f is associated with a real number z_{F_f}, $F_f \in E_f$. The set of all such numbers for all possible values of indexes $j_{k_1}, j_{k_2}, \dots, j_{k_t}$ is called (as in [8]) the t-index matrix and denoted by $\{z_{j_{k_1}, j_{k_2}, \dots, j_{k_t}}\} = \{z_{F_f}\}$. Let $\overline{f} = N(s)\backslash f$. Then $F_{N(s)} = F_f F_{\overline{f}}$ denotes the s-index set $(k_1, k_2, \dots, k_t, k_{t+1}, \dots, k_s)$. Moreover, we assume that if $f = \varnothing$, then E_f contains a special 0-index $F_\varnothing$, where $F_{N(s)} = F_\varnothing F_{N(s)}$. We use the notation

$$\sum_{F_f\in E_f} z_{F_f F_{\overline{f}}} = \sum_{j_{k_1}\in J_{k_1}} \sum_{j_{k_2}\in J_{k_2}} \cdots \sum_{j_{k_t}\in J_{k_t}} z_{F_f F_{\overline{f}}}, \quad F_{\overline{f}} \in E_{\overline{f}}.$$

Then the distribution of a homogeneous resource in hierarchical systems is formulated as follows: for a given set M, $M \subseteq 2^{N(s)}$, find an s-index matrix $\{x_F\}$ satisfying the system of linear multiindex algebraic transport-type inequalities

$$a_{F_{\overline{f}}} \le \sum_{F_f\in E_f} x_{F_f F_{\overline{f}}} \le b_{F_{\overline{f}}}, \quad F_{\overline{f}} \in E_{\overline{f}}, \quad f \in M, \tag{1}$$

with regard for the optimality criteria defined by the vector function $\overrightarrow{Q}(\{x_F\})$ on the set of all s-index matrices with values in R^m. This problem is a many-criteria problem, whose type depends on the choice of the functions $\overrightarrow{Q}(\{x_F\})$. For resource distribution with regard for economic indexes, the functions $\overrightarrow{Q}(\{x_F\})$ can be taken to be linear functions. Then resource distribution in multilevel hierarchical systems is formulated as a many-criteria multiindex linear programming problem. Using the linear convolution of optimality criteria, we can reduce the initial problem to a multiindex linear programming problem under transport-type constraints (1) and criterion

$$F(\{x_F\}) = \sum_{F\in E_{N(s)}} c_F x_F \to \min, \tag{2}$$

which in what follows is denoted by $W(M)$.

In general, such problems can be solved only by universal solution methods for linear programming problems. Since our problem is specific (transport-type linear constraints), effective solution algorithms (different from general tedious solution methods) can be applied to a particular class of these problems.

3. A SOLUTION ALGORITHM

We only study problems $W(M)$ reducible to the L-search for the minimal-cost circulation [9]. In constraints (1), let $a_{F_{\overline{f}}}$ and $b_{F_{\overline{f}}}$ be integers and let $F_{\overline{f}} \in E_{\overline{f}}$, $f \in M$. According to [10]), a problem $W(M)$ is reducible to an L problem if some subset of the components of the optimal solution of the L problem is an optimal solution of the $W(M)$ problem, and the $W(M)$ problem has no admissible solution if the L problem has no admissible solution. Since constraints (1) of the $W(M)$ problem are defined by a set M, $M \subseteq 2^{N(s)}$, we shall determine sets M for which the $W(M)$ problem is reducible to an L problem. Sufficient conditions for the reduction of these problems are formulated in [10], but these conditions cannot be easily verified through the properties of the matrices of constraints. In this paper, we formulate sufficient conditions for reducing the $W(M)$ problem to an L problem in terms of the set M, whose power is one order less than the number of rows in the system of constraints for the $W(M)$ problem.

Theorem 1. *For the $W(M)$ problem to be reducible to an L problem, it is sufficient that there exist a decomposition of the set M into two subsets*

$$M_1 = \left\{ f_1^{(1)}, f_2^{(1)}, \ldots, f_{m_1}^{(1)} \right\} \text{ and } M_2 = \left\{ f_1^{(2)}, f_2^{(2)}, \ldots, f_{m_2}^{(2)} \right\}$$

such that $f_i^{(1)} \subseteq f_{i+1}^{(1)}$, $i = \overline{1, m_1 - 1}$, and $f_i^{(2)} \subseteq f_{i+1}^{(2)}$, $i = \overline{1, m_2 - 1}$.

If the conditions of Theorem 1 hold, then a transport network with two-sided arc capacities is constructed for the multiindex linear programming problem $W(M)$. The admissible circulation is determined by constructing a transport network and solving the maximal flow problem for it. The modified label placement algorithm for the maximal flow problem [11] is of computational complexity $O(n^3)$, where n is the number of vertices in the transport network. The number of vertices in the constructed transport network is $Q = |E_{N(s)}| + \sum_{f \in M} |E_{\overline{f}}|$, i.e., the sum of the number of variables and constraints in the $W(M)$ problem. Under the conditions of Theorem 1, we have $Q \leq 4|E_{N(s)}|(1 - 2^{-s})$. Therefore, for the $W(M)$ problem to be consistent, it is necessary its order be $O(|E_{N(s)}|^3)$ arithmetical operations. The L problem is solved by determining the minimal-cost flow of a given magnitude (for example, with the dual algorithm [12], whose computational complexity is of order n^4). Hence the $W(M)$ can be solved with $O(|E_{N(s)}|^4)$ arithmetical operations.

Whether the conditions of Theorem 1 hold or not is verified by examining all variants since $|M| \leq 2s$ and the set M cannot contain more than two elements of identical power by this theorem.

Theorem 2. *The matrix corresponding to constraint system* (1) *satisfying Theorem 1 is absolutely unimodular.*

By Theorem 2, the solution of the initial resource distribution problem is an integer under the conditions of Theorem 1.

Remark. The conditions of Theorem 1 hold For the problems of transport with intermediate points and production scheduling, i.e., these problems are reducible to the corresponding minimal-cost circulation problems. Indeed, for the transport problem, we have $M = \{\{j,k\}, \{i,k\}, \{i\}, \{k\}, \varnothing\}$ and there exists a decomposition into subsets $M_1 = \{\{j,k\}, \{k\}, \varnothing\}$ and $M_2 = \{\{i,k\}, \{i\}\}$ satisfying the conditions of Theorem 1. For the production scheduling problem, $M = \{\{i,j,k\}, \{i,k,t\}, \{j,k\}, \{k,t\}, \{t\}, \varnothing\}$ and its decomposition into subsets $M_1 = \{\{i,j,k\}, \{j,k\}, \varnothing\}$ and $M_2 = \{\{i,k,t\} \{k,t\}, \{t\}\}$, satisfies the conditions of Theorem 1. For the problem of distribution of capacities of data transmission channels described in the Introduction, $M = \{\{j,k\}, \{i,k\}, \{i,j\}, \varnothing\}$ and there is no decomposition of this set satisfying the conditions of Theorem 1. Moreover, the absolute unimodularity conditions are not satisfied for the matrix of constraints of this problem and, therefore, this problem cannot be reduced to an L problem with integral initial data.

4. REALIZATION OF THE ALGORITHM

The existing software packages, such as the Microsoft Business Solutions-Navision [13], Preactor [14], and Quintiq [15], for the resource distribution problem are designed for decision making in production systems, viz., planning, control, budgeting, etc. Solution of multiindex resource distribution problems by our method and the results thus obtained are based on an interactive program in C++ with MFC library and MS Visual C++.NET compiler.

The main routines of our package are

- design and determination of parameters of the elements of a hierarchical system,
- random generation of hierarchical systems with given parameters, and
- solution by different algorithms.

Our package includes an algorithm for determining deadend flows [11] (for finding the initial flow) and an algorithm for finding the deadend flow with regard for cost indexes (for realizing a modified algorithm for finding minimal-cost flow of a given magnitude), a label placement routine for the maximal-flow problem [11], a modified label placement method (with a laminated network) for the maximal-flow problem [11], a reversed algorithm for determining the minimal-cost flow of a given magnitude [11], and a Bellman–Ford–Moore algorithm for finding negative-cost cycles [16].

Using the initial data of a problem, our system can verify whether the conditions of Theorem 1 hold or not, construct the corresponding transport network, find the minimal-cost circulation for this network, and thereby find the optimal solution of the initial problem. The optimal solution is analyzed to find the "critical" elements (elements for which the resource distribution is close to the boundary of the constraint segment), resources are redistributed with regard for the constraints related to critical elements in interactive mode. This package is helpful in solving resource distribution problems for hierarchical systems containing up to 100 000 elements.

5. AN EXAMPLE

Let us consider a three-index linear programming problem $W(M)$ under transport-type constraints with indexes i, j, k, $i \in I$, $j \in J$, $k \in K$ for a given set $M = \{\{i,k\},\{j,k\},\{k\},\varnothing\}$:

$$a_j \le \sum_{i\in I}\sum_{k\in K} x_{ijk} \le b_j, \quad \in J,$$

$$c_i \le \sum_{j\in J}\sum_{k\in K} x_{ijk} \le d_i, \quad i \in I,$$

$$g_{ij} \le \sum_{k\in K} x_{ijk} \le h_{ij}, \quad i \in I,\ j \in J,$$

$$r_{ijk} \le x_{ijk} \le s_{ijk}, \quad i \in I, \quad j \in J, \quad k \in k,$$

and criterion

$$\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ijk}x_{ijk} \to \min.$$

The decomposition of the set M into $M_1 = \{\{i,k\},\{k\},\varnothing\}$ and $M_2 = \{\{j,k\}\}$ satisfies the conditions of Theorem 1, i.e., the initial problem is reducible to a minimal-cost circulation problem.

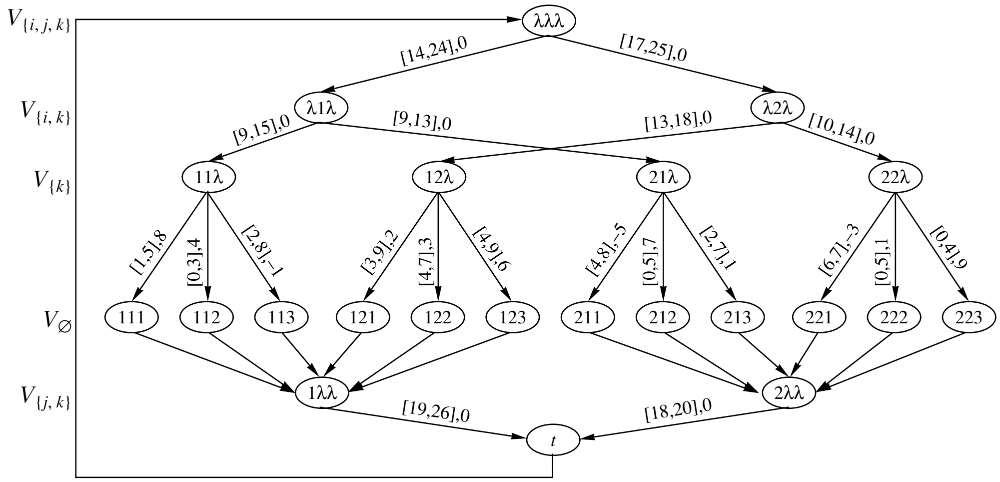

Figure.

Let $I = \{1, 2\}$, $J = \{1, 2\}$, and $K = \{1, 2, 3\}$ and let the $W(M)$ problem be of the form

$$14 \leq \sum_{i=1}^{2}\sum_{k=1}^{3} x_{i1k} \leq 24, \quad 17 \leq \sum_{i=1}^{2}\sum_{k=1}^{3} x_{i2k} \leq 25,$$

$$19 \leq \sum_{j=1}^{2}\sum_{k=1}^{3} x_{1jk} \leq 26, \quad 18 \leq \sum_{j=1}^{2}\sum_{k=1}^{3} x_{2jk} \leq 20,$$

$$9 \leq \sum_{k=1}^{3} x_{11k} \leq 15, \quad 13 \leq \sum_{k=1}^{3} x_{12k} \leq 18,$$

$$9 \leq \sum_{k=1}^{3} x_{21k} \leq 13, \quad 10 \leq \sum_{k=1}^{3} x_{22k} \leq 14,$$

$$1 \leq x_{111} \leq 5, 0 \leq x_{112} \leq 3, 2 \leq x_{113} \leq 8, 3 \leq x_{121} \leq 9,$$

$$4 \leq x_{122} \leq 7, 4 \leq x_{123} \leq 9, 4 \leq x_{211} \leq 8, 0 \leq x_{212} \leq 5,$$

$$2 \leq x_{213} \leq 7, 6 \leq x_{221} \leq 7, 0 \leq x_{222} \leq 5, \quad 0 \leq x_{223} \leq 4,$$

$$8x_{111}+4x_{112}-x_{113}+2x_{121}+3x_{122}+6x_{123}-5x_{211}+7x_{212}+x_{213}-3x_{221}+x_{222}+9x_{223} \to \max.$$

The transport network for the $W(M)$ problem is shown in the figure. Segments corresponding to capacities and costs are joined by arcs. Arcs for which segments are not shown have zero lower capacity bound, unbounded upper capacity bound, and zero cost. The character λ shows that summation is taken over all values of the index at the corresponding corner point.

Table 1 lists the admissible circulation defined by the magnitude of flows on arcs. In this table, the start and end of an arc are shown by u and v, respectively, and the flow magnitude on arc (u, v) by $G(u, v)$.

An admissible circulation defines the consistency of constraints of the corresponding $W(M)$ problem, whose admissible solution is shown in Table 2.

Table 3 shows minimal-cost circulations.

The solution defined by the minimal-cost circulation of the $W(M)$ problem is shown in Table 4.

Table 1

u	$(\lambda,\lambda,\lambda)$	$(\lambda,\lambda,\lambda)$	$(\lambda,1,\lambda)$	$(\lambda,1,\lambda)$	$(\lambda,2,\lambda)$	$(\lambda,2,\lambda)$
v	$(\lambda,1,\lambda)$	$(\lambda,2,\lambda)$	$(1,1,\lambda)$	$(2,1,\lambda)$	$(1,2,\lambda)$	$(2,2,\lambda)$
$G(u,v)$	18	23	9	9	13	10
u	$(1,1,\lambda)$	$(1,1,\lambda)$	$(1,1,\lambda)$	$(1,2,\lambda)$	$(1,2,\lambda)$	$(1,2,\lambda)$
v	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
$G(u,v)$	5	5	5	3	4	6
u	$(2,1,\lambda)$	$(2,1,\lambda)$	$(2,1,\lambda)$	$(2,2,\lambda)$	$(2,2,\lambda)$	$(2,2,\lambda)$
v	$(2,1,1)$	$(2,1,2)$	$(2,1,3)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
$G(u,v)$	6	0	3	7	0	3
u	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
v	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$
$G(u,v)$	5	2	2	3	4	6
u	$(2,1,1)$	$(2,1,2)$	$(2,1,3)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
v	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$
$G(u,v)$	6	0	3	7	0	3
u	$(1,\lambda,\lambda)$	$(2,\lambda,\lambda)$	t	–	–	–
v	t	t	$(\lambda,\lambda,\lambda)$	–	–	–
$G(u,v)$	22	19	41	–	–	–

Table 2

x_{111}	x_{112}	x_{113}	x_{121}	x_{122}	x_{123}	x_{211}	x_{212}	x_{213}	x_{221}	x_{222}	x_{223}
5	2	2	3	4	6	6	0	3	7	0	3

Table 3

u	$(\lambda,\lambda,\lambda)$	$(\lambda,\lambda,\lambda)$	$(\lambda,1,\lambda)$	$(\lambda,1,\lambda)$	$(\lambda,2,\lambda)$	$(\lambda,2,\lambda)$
v	$(\lambda,1,\lambda)$	$(\lambda,2,\lambda)$	$(1,1,\lambda)$	$(2,1,\lambda)$	$(1,2,\lambda)$	$(2,2,\lambda)$
$G(u,v)$	20	25	10	10	15	10
u	$(1,1,\lambda)$	$(1,1,\lambda)$	$(1,1,\lambda)$	$(1,2,\lambda)$	$(1,2,\lambda)$	$(1,2,\lambda)$
v	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
$G(u,v)$	5	3	2	3	4	8
u	$(2,1,\lambda)$	$(2,1,\lambda)$	$(2,1,\lambda)$	$(2,2,\lambda)$	$(2,2,\lambda)$	$(2,2,\lambda)$
v	$(2,1,1)$	$(2,1,2)$	$(2,1,3)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
$G(u,v)$	4	4	2	6	0	4
u	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,2,1)$	$(1,2,2)$	$(1,2,3)$
v	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$	$(1,\lambda,\lambda)$
$G(u,v)$	5	3	2	3	4	8
u	$(2,1,1)$	$(2,1,2)$	$(2,1,3)$	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$
v	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$	$(2,\lambda,\lambda)$
$G(u,v)$	4	4	2	6	0	4
u	$(1,\lambda,\lambda)$	$(2,\lambda,\lambda)$	t	–	–	–
v	t	t	$(\lambda,\lambda,\lambda)$	–	–	–
$G(u,v)$	25	20	45	–	–	–

Table 4

x_{111}	x_{112}	x_{113}	x_{121}	x_{122}	x_{123}	x_{211}	x_{212}	x_{213}	x_{221}	x_{222}	x_{223}
5	3	2	3	4	8	4	4	2	6	0	4

6. CONCLUSIONS

Our approach to study multiindex resource distribution problem in hierarchical systems is helpful in applying effective flow algorithms for a wide range of applied problems. The software package based on our results was tested by solving production planning problems for single- and short-run production plants and in problems of distribution of finances to various departments of a scientific research institute with an experimental production unit.

APPENDIX

Proof of Theorem 1 is based on the construction of a network model. Let us introduce the following: The value set of indexes $F_{f^{(2)}} = \left(j_{k_1^{(2)}}, j_{k_2^{(2)}}, \ldots, j_{k_{t_2}^{(2)}}\right)$ contains the value set of indexes $F_{f^{(1)}} = \left(j_{k_1^{(1)}}, j_{k_2^{(1)}}, \ldots, j_{k_{t_1}^{(1)}}\right)$, denoted by $F_{f^{(1)}} \prec F_{f^{(2)}}$, if $f^{(1)} \subseteq f^{(2)}$ and $j_{k_{i_1}^{(1)}} = j_{k_{i_2}^{(2)}}$ for $k_{i_1}^{(1)} = k_{i_2}^{(2)}$. It is assumed that $F_\varnothing \prec F_{f'}$ for any $f' \subseteq N(s)$. Without loss of generality, we assume that $f_1^{(1)} = \varnothing$ and $f_{m_1}^{(1)} = \{1, 2, \ldots, s\}$ if $\varnothing$, $N(s) \in M$. If some of these sets is not contained in M, then it is added, assuming that the corresponding a_{F_f} are zero and b_{F_f} are infinitely large. For the sake of convenience, we take $f_0^{(2)} = f_1^{(1)}$.

We now construct a network consisting of

(1) nodes of the set $V_{f_i^{(j)}} = \left\{v_{F_{\overline{f_i^{(j)}}}} \,|\, F_{\overline{f_i^{(j)}}} \in E_{f_i^{(j)}}\right\}$, $i = \overline{1, m_j}$, $j = \overline{1, 2}$,

(2) a special closure node t,

(3) arcs $\left(v_{F_{\overline{f_{i+1}^{(1)}}}}, v_{F_{\overline{f_i^{(1)}}}}\right)$ defined by the set M_1 with lower capacity bound $a_{F_{\overline{f_i^{(1)}}}}$ and upper capacity bound $b_{F_{\overline{f_i^{(1)}}}}$ if $F_{\overline{f_{i+1}^{(1)}}} \prec F_{\overline{f_i^{(1)}}}$, where $F_{\overline{f_{i+1}^{(1)}}} \in E_{\overline{f_{i+1}^{(1)}}}$ and $F_{\overline{f_i^{(1)}}} \in E_{\overline{f_i^{(1)}}}$, $i = \overline{1, m_1 - 1}$,

(4) arcs $\left(t, v_{F_{\overline{f_{m_1}^{(1)}}}}\right)$ defined by the set M_1 with lower capacity bound $a_{F_{\overline{f_{m_1}^{(1)}}}}$ and upper capacity bound $b_{F_{\overline{f_{m_1}^{(1)}}}}$, where $F_{\overline{f_{m_1}^{(1)}}} \in E_{\overline{f_{m_1}^{(1)}}}$,

(5) arcs $\left(v_{F_{\overline{f_i^{(2)}}}}, v_{F_{\overline{f_{i+1}^{(2)}}}}\right)$ defined by the set M_2 with lower capacity bound $a_{F_{\overline{f_i^{(2)}}}}$ and upper capacity bound $b_{F_{\overline{f_i^{(2)}}}}$ if $F_{\overline{f_{i+1}^{(2)}}} \prec F_{\overline{f_i^{(2)}}}$, where $F_{\overline{f_i^{(2)}}} \in E_{\overline{f_i^{(2)}}}$ and $F_{\overline{f_{i+1}^{(2)}}} \in E_{\overline{f_{i+1}^{(2)}}}$, $i = \overline{1, m_2 - 1}$,

(6) arcs $\left(v_{F_{\overline{f_{m_2}^{(2)}}}}, t\right)$ defined by the set M_2 with lower capacity bound $a_{F_{\overline{f_{m_2}^{(2)}}}}$ and upper capacity bound $b_{F_{\overline{f_{m_2}^{(2)}}}}$, where $F_{\overline{f_{m_2}^{(2)}}} \in E_{\overline{f_{m_2}^{(2)}}}$, and

(7) a closure arc $\left(v_{F_{\overline{f_1^{(1)}}}}, v_{F_{\overline{f_1^{(2)}}}}\right)$ with zero lower capacity bound and unbounded upper capacity bound if $F_{\overline{f_1^{(2)}}} \prec F_{\overline{f_1^{(1)}}}$, where $F_{\overline{f_1^{(1)}}} \in E_{\overline{f_1^{(1)}}}$, $F_{\overline{f_1^{(2)}}} \in E_{\overline{f_1^{(2)}}}$.

Arcs of the type $\left(v, v_{F_{\overline{f_1^{(1)}}}}\right)$ are associated with costs $c_{F_{\overline{f_1^{(1)}}}}$, where v is an arbitrary node of the network $F_{\overline{f_1^{(1)}}} \in E_{\overline{f_1^{(1)}}}$, and other arcs are associated with zero costs.

For every admissible solution of the $W(M)$ problem, there obviously exists an admissible circulation in the network and vice versa (here every variable x_F of the $W(M)$ problem is associated with an arc (v, v_F), $F \in E_{N(s)}$). Furthermore, the cost of an admissible circulation and value of the aim function (2) coincide on the corresponding solution. The system of constraints (1) is compatible if and only if the network contains an admissible circulation. Hence this construct reduces the $W(M)$ problem to an L problem.

Proof of Theorem 2. We demonstrate this theorem via *reductio ad absurdum*. Assume that constraints (1) satisfy the conditions of Theorem 1, but the matrix corresponding to them is not absolutely unimodular. Since the absolute unimodularity conditions for integral $a_{F_{\overline{f}}}$ $b_{F_{\overline{f}}}$, $F_{\overline{f}} \in E_{\overline{f}}$, $f \in M$, are necessary and sufficient for all vertices corresponding to the polyhedron of constraints of system (1) to be integral numbers (see [17]), we can construct a linear aim function such that the $W(M)$ problem has a unique optimal integral solution. By Theorem 1, the $W(M)$ problem is reduced to an L problem, and the minimal-cost circulation problem, by the integral flow theorem (see [9]), always has an optimal integral solution (if the problem has a solution). This contradiction demonstrates the validity of the theorem.

REFERENCES

1. Voronin, A.A. and Mishin, S.P., *Optimal'nye ierarkhicheskie struktury* (Optimal Hierarchical Structures), Moscow: Inst. Probl. Upravlen., 2003.
2. Elsasser, R., Monien, B., and Preis, R., Diffusion Schemes for Load Balancing on Heterogeneous Networks, *Theory Comput. Syst.*, 2002, vol. 35, no. 3, pp. 305–320.
3. Gairing, M., Lucking, T., Mavronicolas, M., and Monien, B., Computing Nash Equilibria for Scheduling on Restricted Parallel Links, *Proc. 36th Annual ACM Sympos. Theory Comput.*, 2004, pp. 613–622.
4. Dunne, P.E., Extremal Behavior in Multiagent Contract Negotiation, *J. Artificial Intelligence Res.*, 2005, vol. 23, no. 1, pp. 41–78.
5. Prilutskii, M.Kh., Distribution of a Homogeneous Resource in Tree-Structured Hierarchical Systems, *Tr. III Int. Conf. Identification of Systems and Control Problems (SICPRO'2000)*, Moscow: Inst. Probl. Upravlen., 2000, pp. 2038–2049.
6. Prilutskii, M.Kh. and Kartomin, ,A.G., Flow Algorithms for Resource Distribution in Hierarchical Systems, *"Issledovano v Rossii,"* 2003, no. 39, pp. 444–452 (http://zhurnal.ape.relarn.ru/articles/2003/039.pdf).
7. Prilutskii, M.Kh., Many-Criteria Distribution of a Homogeneous Resource in Hierarchical Systems, *Avtom. Telemekh.*, 1996, no. 2, pp. 24–29.
8. Raskin, L.T. and Kirichenko, I.O., *Mnogoindeksnye zadachi lineinogo programmirovaniya* (Multiindex Linear Programming Problems), Moscow: Radio i Svyaz', 1982.
9. Ford, L.R., Jr. and Fulkerson, D.R., *Flows in Networks*, Princeton: Princeton Univ. Press, 1962. Translated under the title *Potoki v setyakh*, Moscow: Mir, 1966.
10. Litvak, B.G. and Rappoport, A.M., Linear Programming Problems admitting a Network Formulation, *Ekonom. Mat. Metody*, 1970, vol. 6, no. 4, pp. 594–604.
11. Papadimitrou, Ch. and Steiglitz, K., *Combinatorial Optimization: Algorithms and Complexity*, Englewood Cliffs: Prentice Hall, 1982. Translated under the title *Kombinatornaya optimizatsiya: Algoritmy i slozhnost'*, Moscow: Mir, 1985.
12. Hu, T.C., *Integer Programming and Network Flows*, Reading: Addison-Wesley, 1969. Translated under the title *Tselochislennoe programmirovanie i potoki v setyakh*, Moscow: Mir, 1974.
13. Manuscript at http://www.microsoft.com/.
14. Manuscript at http://www.preactor.com/.
15. Manuscript at http://www.quintiq.com/.
16. Cherkassky, B.V. and Goldberg, A.V., Negative-Cycle Detection Algorithms, *Proc. 4th Eur. Sympos. Algorithms*, 1996, pp. 349–363.
17. Gofman, A.G. and Kraskal, G.B., *Integer Boundary Points of Convex Polyhedrons*, in *Lineinye neravenstva i smezhnye voprosy* (Linear Inequalities and Related Problems), Moscow: Inostrannaya Literatura, 1959.

This paper was recommended for publication by P.Yu. Chebotarev, a member of the Editorial Board